HOMOLOGY AND COHOMOLOGY

by

© ERIC G. CHISLETT

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "HOMOLOGY AND COHOMOLOGY", submitted by ERIC G. CHISLETT in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

This thesis is devoted to a study of homology and cohomology as defined by complexes of modules (or just abelian groups). Although homology theories are now usually defined axiomatically, their existence is proven by construction; that is by obtaining from a group, topological space, algebra, etc., a complex from which we obtain the homology and cohomology groups of the original structure.

Chapters I and II give the essential features from homological algebra, category theory and the homology of complexes necessary for the development of any homology theory. Chapters III and IV constitute the main body of this thesis: In them the functors Ext and Tor are developed and studied in relation to the cohomology and homology groups, respectively. As an example, the cohomology of groups is studied in some detail in Chapter V and is used in Chapter VI to prove Schur's Theorem and to view briefly Spaces with Operators.

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TABLE OF CONTENTS

CHAPTER I

BASIC PROPERTIES

In this chapter we develop, without proof, those notions from Homological Algebra and from Category Theory essential for the development of homology and cohomology. In so doing, many very interesting aspects of the theory, in particular some of the proofs of the stated theorems, are omitted. Throughout this chapter no reference is made to any specific text as details can be found in most standard texts on the subject.

1. Modules and Homomorphisms

Let R be a ring with unity $1 \neq 0$. A *left R-module* A is an additive abelian group A together with an operation $ra \in A$, for $r \in A$, $a \in A$, such that

$$r(a + a') = ra + ra' , \quad (rr')a = r(r'a) ,$$
$$(r + r')a = ra + r'a , \quad 1a = a .$$

We shall frequently use *right R-modules*, which are abelian groups A with $ar \in A$ defined so as to satisfy the corresponding four identities.

In the special case $R = Z$ the ring of integers, the R-modules are simply abelian groups; if $R = Z_k$, the ring of integers modulo k,

they are abelian groups in which every element has order a divisor of k ; if R is a field, they are vector spaces over R . The ring R is itself a left and a right R-module.

A subset S of an R-module A is a *submodule* $(S \subseteq A)$ if S is closed under addition and $r \in R$, $s \in S$ implies $rs \in S$; then S is itself an R-module. In this case the quotient group A/S is an R-module with module multiplication given by $r(a + S) = ra + S$. A/S is termed the *quotient module* of A by S .

Given two R-modules A and B , an R-module homomorphism of A into B is a function $\alpha : A \to B$ such that

$$\alpha(a + a') = \alpha a + \alpha a' \ , \ \alpha(ra) = r(\alpha a)$$

for a , $a' \in A$, $r \in R$. The *kernel* of α $\operatorname{Ker} \alpha = \{a \in A \ ; \ \alpha a = 0\}$ and *image* of α $\operatorname{Im} \alpha = \{\alpha a \in B \ ; \ a \in A\}$ are submodules of A and B respectively. An R-module homomorphism $\alpha : A \to B$ is called an *epimorphism, monomorphism* or *isomorphism,* according as $\operatorname{Im} \alpha = B$, $\operatorname{Ker} \alpha = 0$, or $\operatorname{Im} \alpha = B$ and $\operatorname{Ker} \alpha = 0$, respectively.

If S is a submodule of A , there is a natural homomorphism $A \to A/S$ given by $a \to a + S$. If $\alpha : A \to B$ is a homomorphism and $S \subseteq A$, $T \subseteq B$ are submodules with the property that $\alpha S \subseteq T$, there is a new homomorphism $\alpha_* : A/S \to B/T$ with $\alpha_*(a + S) = \alpha a + T$. α_* is said to be *induced* by α .

If α_1 , $\alpha_2 : A \to B$ are homomorphisms, then their *sum* $\alpha_1 + \alpha_2$

with $(\alpha_1 + \alpha_2)a = \alpha_1 a + \alpha_2 a$ is also a homomorphism. If $\alpha : A \to B$ and $\beta : B \to C$ are homomorphisms, then the *composite* function $\beta\alpha : A \to C$ with $(\beta\alpha)a = \beta(\alpha a)$ is a homomorphism.

A *left inverse* of $\alpha : A \to B$ is a homomorphism $\beta : B \to A$ with $\beta\alpha = 1_A$. If β is also a right inverse, it is unique and α is an isomorphism.

2. Exact Sequences and Direct Sums

A sequence of R-modules and homomorphisms

$$\cdots \to A_{i+1} \xrightarrow{\alpha_{i+1}} A_i \xrightarrow{\alpha_i} A_{i-1} \to \cdots$$

is said to be *exact* if $\operatorname{Im} \alpha_{i+1} = \operatorname{Ker} \alpha_i$, for all i. A short exact sequence

$$0 \to A \xrightarrow{\kappa} B \xrightarrow{\sigma} C \to 0$$

has κ a monomorphism and σ an epimorphism. We write $E = (\kappa,\sigma) : A \rightarrowtail B \twoheadrightarrow C$.

Each homomorphism $\alpha : A \to B$ gives two quotient modules

$$\operatorname{Coim} \alpha = A/\operatorname{Ker} \alpha \quad , \quad \operatorname{Coker} \alpha = B/\operatorname{Im} \alpha$$

called the *coimage* and *cokernel* of α. This definition gives two

short exact sequences

$$\operatorname{Ker}\alpha \rightarrowtail A \twoheadrightarrow \operatorname{Coim}\alpha\ ,\ \operatorname{Im}\alpha \rightarrowtail B \twoheadrightarrow \operatorname{Coker}\alpha\ .$$

If
$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\theta"'] & B \arrow[d, "\phi"] \\ A' \arrow[r, "\beta"] & B' \end{tikzcd}$$
and
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

are diagrams of modules and homomorphisms, we say they are *commutative* if in the first $\beta\theta = \phi\alpha$ while in the second $gf = h$. Larger diagrams are commutative if each small square or triangle commutes.

Two frequently used lemmas follow.

Lemma 2.1. (The Short Five Lemma) Let the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

have exact rows. Then if α and γ are isomorphisms, monomorphisms or epimorphisms, respectively, then β is an isomorphism, monomorphism or epimorphism, respectively.

Lemma 2.2. (The Five Lemma) Let a commutative diagram

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\alpha_1"'] & \cdot \arrow[r] \arrow[d, "\alpha_2"'] & \cdot \arrow[r] \arrow[d, "\alpha_3"'] & \cdot \arrow[r] \arrow[d, "\alpha_4"'] & \cdot \arrow[d, "\alpha_5"'] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

have exact rows. If α_1, α_2, α_4 and α_5 are isomorphisms, then so is α_3.

The *direct sum* $A_1 \oplus A_2$ of two R-modules A_1 and A_2 is the R-module consisting of all ordered pairs (a_1,a_2) for $a_i \in A_i$, with operations given by

$$(a_1,a_2) + (a_1',a_2') = (a_1+a_1',a_2+a_2') \ , \quad r(a_1,a_2) = (ra_1,ra_2) \ .$$

The functions ι and π defined by $\iota_1 a_1 = (a_1,0)$, $\iota_2 a_2 = (0,a_2)$, $\pi_1(a_1,a_2) = a_1$ and $\pi_2(a_1,a_2) = a_2$ are homomorphisms

$$A_1 \underset{\pi_1}{\overset{\iota_1}{\rightleftarrows}} A_1 \oplus A_2 \underset{\pi_2}{\overset{\iota_2}{\leftrightarrows}} A_2 \qquad (2.1)$$

satisfying the identities

$$\pi_1\iota_1 = 1_{A_1} \qquad \pi_1\iota_2 = 0$$

$$\pi_2\iota_2 = 1_{A_2} \qquad \pi_2\iota_1 = 0$$

$$\iota_1\pi_1 + \iota_2\pi_2 = 1_{A_1 \oplus A_2} \ . \qquad (2.2)$$

Call ι_1,ι_2 the injections and π_1,π_2 the projections.

Proposition 2.3. For given modules A_1 and A_2 any diagram

$$A_1 \underset{\pi_1}{\overset{\iota_1}{\rightleftarrows}} B \underset{\pi_2}{\overset{\iota_2}{\leftrightarrows}} A_2$$

of the form (2.1) and satisfying the five identities (2.2) is isomorphic to the direct sum diagram, that is $B \cong A_1 \oplus A_2$.

Proposition 2.4. The following properties of a short exact sequence $(\iota, \pi) : A_1 \rightarrowtail B \twoheadrightarrow A_2$ are equivalent.

i. π has a right inverse.

ii. ι has a left inverse.

iii. The sequence is isomorphic to $A_1 \rightarrowtail A_1 \oplus A_2 \twoheadrightarrow A_2$.

A short exact sequence with one (and hence all) of these properties is said to *split*.

3. Projective and Injective Modules

Let F be an R-module and T a subset of F . We say that F is *free* on T if every $x \in F$ can be written uniquely as a finite sum $\Sigma\, r_i x_i$, for $r_i \in R$ and $x_i \in T$. If T is any set we may define a module F as the set of all finite sums $\Sigma\, r_i x_i$. By identifying $x \in T$ with $1x \in F$, F is free with *base* T . This gives us

Proposition 3.1. Every module is isomorphic to the quotient of a free module.

A module P is called *projective* if for each epimorphism $\sigma : B \twoheadrightarrow C$, any homomorphism $\gamma : P \to C$ can be lifted to a homomorphism $\beta : P \to B$ such that $\sigma\beta = \gamma$. In other words, the

diagram

$$\begin{tikzcd} & P \arrow[dl, dotted] \arrow[d, "\gamma"] \\ B \arrow[r, two heads, "\sigma"'] & C \end{tikzcd}$$

can, by means of the dotted arrow, be made commutative.

Lemma 3.2. Every free module is projective.

Theorem 3.3. (The Characterization of Projectives) The following properties of a module P are equivalent.

i. P is projective.

ii. P is a direct summand of a free module.

iii. Every exact sequence $A \rightarrowtail B \twoheadrightarrow P$ splits.

Corollary 3.4. A direct summand of modules is projective if and only if each summand is projective.

A module Q is called *injective* if given any module B and submodule A, any homomorphism $A \to Q$ can be extended to a homomorphism $B \to Q$. In the language of diagrams this means that every diagram

$$\begin{tikzcd} & Q \\ A \arrow[ur] \arrow[r, tail] & B \arrow[u, dashed] \end{tikzcd}$$

can be completed to a commutative diagram by means of the dotted arrow.

Theorem 3.5. Every module is a submodule of an injective module.

4. Categories and Functors

A category consists of "objects" and "morphisms" which may sometimes be composed. Formally, a *category* $\mathcal{C}$ is a class of *objects* A , B , C , ... together with

i. A family of sets $H(A,B)$, one for each pair of objects, whose elements are called *morphisms* of the object A to the object B . For $\alpha \in H(A,B)$ we write $\alpha : A \to B$.

ii. For each triple of objects A , B , C and morphisms $\alpha \in H(A,B)$, $\beta \in H(B,C)$ there exists a uniquely defined product $\beta\alpha \in H(A,C)$. Furthermore, this product is *associative* ; if we have $\alpha : A \to B$, $\beta : B \to C$, $\gamma : C \to D$, then $(\gamma\beta)\alpha = \gamma(\beta\alpha)$.

iii. A map assigning to each object A a morphism $e_A \in H(A,A)$, called the *identity*. Furthermore, if $\alpha \in H(A,B)$, then $e_B\alpha = \alpha = \alpha e_A$.

Clearly the identity morphism is unique. A morphism $\alpha : A \to B$ is called an *equivalence* in $\mathcal{C}$ if there is in $\mathcal{C}$ another morphism $\beta : B \to A$ such that $\beta\alpha = e_A$ and $\alpha\beta = e_B$. Then β is unique and termed the *inverse* of α . In such a case A and B are said to be *equivalent*.

We list some example of categories.

i. The category of modules and module homomorphisms $\mathcal{C}$ (modules).

ii. The category of topological spaces and continuous maps $\mathcal{C}$ (top spaces).

iii. The category of finite sets and set functions $\mathcal{C}$ (sets).

iv. Given a category $\mathcal{C}$, there is an opposite category $\mathcal{C}^*$ whose objects are the same as those of $\mathcal{C}$ and whose morphisms are in one-one correspondence with those of $\mathcal{C}$, that is $\alpha^* \varepsilon H(B,A)$ in $\mathcal{C}^*$ if and only if $\alpha \varepsilon H(A,B)$ in $\mathcal{C}$. $\alpha^*\beta^*$ is defined and is $(\beta\alpha)^*$ exactly when $\beta\alpha$ is defined.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *covariant (contravariant) functor* T on $\mathcal{C}$ to $\mathcal{D}$ is a pair of functions: An "object function" which assigns to each object $C \varepsilon \mathcal{C}$ an object $T(C) \varepsilon \mathcal{D}$, and a "mapping function" which assigns to each morphism $\gamma : C \to C'$ in $\mathcal{C}$ a morphism $T(\gamma) : T(C) \to T(C')(T(\gamma) : T(C') \to T(C))$ in $\mathcal{D}$. This pair of functions must satisfy the two conditions

$$T(1_C) = 1_{T(C)}$$

$$T(\beta\gamma) = T(\beta)T(\gamma) \quad (T(\beta\gamma) = T(\gamma)T(\beta))$$

for $C \varepsilon \mathcal{C}$ and $\beta\gamma$ defined.

For example there is a covariant functor from $\mathcal{C}$ (sets) to $\mathcal{C}$ (modules) which assigns to every set the R-module generated by it. For any category $\mathcal{C}$ there is a contravariant functor to its opposite category $\mathcal{C}^*$ which assigns to an object $C \varepsilon \mathcal{C}$, the same object $C \varepsilon \mathcal{C}^*$, and to a morphism $\gamma : C \to C'$ in $\mathcal{C}$, the morphism $\gamma^* : C' \to C$ in $\mathcal{C}^*$.

A *natural transformation* $h : S \to T$ between two covariant functors on $\mathscr{C}$ to $\mathscr{D}$ is a function which assigns to every object $C \in \mathscr{C}$ a morphism $h(C) : S(C) \to T(C)$ of $\mathscr{D}$ such that for each $\gamma : C \to C'$ in $\mathscr{C}$ the diagram

$$\begin{array}{ccc} S(C) & \xrightarrow{h(C)} & T(C) \\ S(\gamma) \downarrow & & T(\gamma)\downarrow \\ S(C') & \xrightarrow{h(C')} & T(C') \end{array}$$

is commutative. A similar definiton is given when S and T are both contravariant or when one is covariant while the other contravariant. When $h(C)$ satisfies the above commutativity condition we say more briefly that "h is natural". If in addition $h(C)$ is an equivalence, we say that h is a *natural isomorphism.*

Functors in several variables may be covariant in some and contravariant in others. As an example, let $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{D}$ be three categories. A *bifunctor* T on $\mathscr{B} \times \mathscr{C}$ to $\mathscr{D}$, contravariant in $\mathscr{B}$ and covariant in $\mathscr{C}$, is a pair of functions: An object function assigning to each $B \in \mathscr{B}$ and $C \in \mathscr{C}$ an object $T(B,C) \in \mathscr{D}$, and a mapping function assigning to morphisms $\beta : B \to B'$ and $\gamma : C \to C'$ a morphism $T(\beta,\gamma) : T(B',C) \to T(B,C')$ in $\mathscr{D}$ satisfying

$$T(1_B,1_C) = 1_{T(B,C)}$$

$$T(\beta'\beta,\gamma'\gamma) = T(\beta,\gamma') \, T(\beta',\gamma)$$

where $\beta'\beta$ and $\gamma'\gamma$ are defined.

A bifunctor T on $\mathcal{B} \times \mathcal{C}$ to $\mathcal{D}$ is said to be *additive* if for $\gamma, \gamma_1, \gamma_2 \in \mathcal{C}$ and $\beta, \beta_1, \beta_2 \in \mathcal{B}$ we have

$$T(\beta_1 + \beta_2, \gamma) = T(\beta_1, \gamma) + T(\beta_2, \gamma)$$
$$T(\beta, \gamma_1 + \gamma_2) = T(\beta, \gamma_1) + T(\beta, \gamma_2) \quad .$$

5. <u>The Functor Hom</u>

Let A and B be R-modules. The set

$$\mathrm{Hom}\ (A,B) = \{f \ ; \ f : A \to B\}$$

of all R-module homomorphisms of A into B is an abelian group under the addition $f,g : A \to B$ $(f + g)a = fa + ga$. In case R is commutative, Hom (A,B) becomes an R-module with $rf : A \to B$ defined for $r \in R$, $f : A \to B$ by $(rf)a = r(fa)$.

For $\alpha : A \to A'$ and $\beta : B \to B'$, each $f : A' \to B$ determines a composite $\beta f \alpha : A \to B'$; the correspondence $f \to \beta f \alpha$ is a homomorphism

$$\mathrm{Hom}\ (\alpha,\beta) : \mathrm{Hom}\ (A',B) \to \mathrm{Hom}\ (A,B')$$

of abelian groups having the properties

$$\mathrm{Hom}\ (1,1) = \text{The identity}$$
$$\mathrm{Hom}\ (\alpha\alpha',\beta\beta') = \mathrm{Hom}\ (\alpha',\beta)\ \mathrm{Hom}\ (\alpha,\beta')$$

$$\text{Hom}\ (\alpha + \alpha', \beta) = \text{Hom}\ (\alpha,\beta) + \text{Hom}\ (\alpha',\beta)$$

$$\text{Hom}\ (\alpha, \beta + \beta') = \text{Hom}\ (\alpha,\beta) + \text{Hom}\ (\alpha,\beta')\ .$$

Hence Hom is an additive bifunctor, contravariant in the first variable and covariant in the second. Furthermore it follows that

$$\text{Hom}\ (A_1 \oplus A_2, B) \cong \text{Hom}\ (A_1,B) \oplus \text{Hom}\ (A_2,B)$$

$$\text{Hom}\ (A, B_1 \oplus B_2) \cong \text{Hom}\ (A,B_1) \oplus \text{Hom}\ (A,B_2)\ .$$

The *dual* of a left R-module A is the right R-module $A^* = \text{Hom}\ (A,R)$ with module structure fr for $f : A \to R$ and $r \in R$ given by $(fr)a = (fa)r$. Thus the dual is an additive contravariant functor with $(A \oplus B)^* \cong A^* \oplus B^*$ from $\mathcal{C}$ (left modules) to $\mathcal{C}$ (right modules).

For R commutative $A^* \cong A$ under $f \longleftrightarrow f(1)$.

Theorem 5.1. For any module M and exact sequence $(\kappa,\sigma) : A \rightarrowtail B \twoheadrightarrow C$, the sequences

$$0 \to \text{Hom}\ (M,A) \xrightarrow{\kappa_*} \text{Hom}\ (M,B) \xrightarrow{\sigma_*} \text{Hom}\ (M,C)$$

$$0 \to \text{Hom}\ (C,M) \xrightarrow{\kappa^*} \text{Hom}\ (B,M) \xrightarrow{\sigma^*} \text{Hom}\ (A,M)$$

are exact where $\kappa_* = \text{Hom}\ (1,\kappa)$, $\sigma_* = \text{Hom}\ (1,\sigma)$, $\kappa^* = \text{Hom}\ (\kappa,1)$ and $\sigma^* = \text{Hom}\ (\sigma,1)$. Furthermore, M is projective if and only if σ_*

is an epimorphism.

The exactness on the right of the second sequence above is completed by

Theorem 5.2. (The Characterization of Injectives) The following properties of an R-module A are equivalent.

i. Q is injective.

ii. Every short exact sequence $Q \rightarrowtail B \twoheadrightarrow C$ splits.

iii. For every monomorphism $\kappa : A \to B$, $\kappa^* : \mathrm{Hom}(B,Q) \to \mathrm{Hom}(A,Q)$ is an epimorphism.

6. The Tensor Product as a Functor

Let G be a right R-module and A a left R-module, usually abbreviated to G_R, ${}_RA$. Their *tensor product* $G \otimes A$ is the abelian group generated by the symbols $g \otimes a$ for $g \in G$, $a \in A$, subject to the relations

$$(g + g') \otimes a = g \otimes a + g' \otimes a \ , \quad g \otimes (a + a') = g \otimes a + g \otimes a'$$

$$gr \otimes a = g \otimes ra \qquad a \in A \ , \ g \in G \ , \ r \in R \ .$$

If $G \times A$ is the caresian product of the modules G_R and ${}_RA$, while M is an arbitrary abelian group, we call a function f on $G \times A$ to M *bilinear* if

$$f(g + g',a) = f(g,a) + f(g',a) \ , \ f(g,a + a') = f(g,a) + f(g,a')$$

$$f(gr,a) = f(g,ra) \ .$$

This gives the following universal property for the tensor product.

Theorem 6.1. Given modules G_R and $_RA$ and a bilinear function f on $G \times A$ to an abelian group M, there is a unique group homomorphism $\omega : G \otimes A \to M$ with $\omega(g \otimes a) = f(g,a)$.

For module homomorphisms $\gamma : G \to G'$ and $\alpha : A \to A'$ there is a group homomorphism $\gamma \otimes \alpha : G \otimes A \to G' \otimes A'$ given by $(\gamma \otimes \alpha)(g \otimes a) = \gamma g \otimes \alpha a$. It follows that the tensor product is an additive covariant functor from $\mathcal{C}$ (modules) to $\mathcal{C}$ (groups) and that

$$G \otimes (A \oplus B) \cong (G \otimes A) \oplus (G \otimes B)$$
$$(G \oplus G') \otimes A \cong (G \otimes A) \oplus (G' \otimes A) .$$

We also have isomorphisms $G \otimes R \cong G$ and $R \otimes A \cong A$ under $g \otimes r \longleftrightarrow gr$ and $r \otimes a \longleftrightarrow ra$, respectively.

If R is a commutative ring, then any right R-module A may be regarded as a left R-module simply by defining ra, with $r \in R$, as ar. Then $(rr')a = r(r'a)$. Thus for modules A and B over a commutative ring R, the tensor product $A \otimes B$ is another R-module with module structure defined by

$$r(a \otimes b) = ra \otimes b \quad (\text{or} = a \otimes rb) .$$

In such a case $A \otimes B \cong B \otimes A$.

We state a fundamental result.

Theorem 6.2. If G is a right R-module, while $(\kappa,\sigma) : A \rightarrowtail B \twoheadrightarrow C$ is an exact sequence of left R-modules, then

$$G \otimes A \xrightarrow{1 \otimes \kappa} G \otimes B \xrightarrow{1 \otimes \sigma} G \otimes C \rightarrow 0$$

is an exact sequence of abelian groups. Furthermore, if G is a free module, $1 \otimes \kappa$ is a monomorphism. There is a corresponding result if the roles of left and right modules are interchanged.

CHAPTER II

HOMOLOGY OF COMPLEXES

This chapter considers the basic concepts of constructing homology and cohomology groups of a chain complex. The basic idea is that a short exact sequence of complexes yields a long exact sequence of homology groups. An illustrating example is provided by considering briefly the singular homology and cohomology groups of a topological space.

1. Complexes

For any ring R, a *chain complex* K of R-modules is a family $\{K_n, \partial_n\}$ of R-modules K_n, and R-module homomorphisms $\partial_n : K_n \to K_{n-1}$

$$\cdots \to K_{n+1} \to K_n \to K_{n-1} \to \cdots$$

defined for all integers, $-\infty < n < \infty$, and such that $\partial_n \partial_{n+1} = 0$, that is $\operatorname{Ker} \partial_n \supseteq \operatorname{Im} \partial_{n+1}$.

The *homology* $H(K)$ of the complex K is the family of modules

$$H_n(K) = \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1} .$$

An *n-chain* of K is an element of K_n while an element of the

submodule $C_n(K) = \mathrm{Ker}\,\partial_n$ is termed an *n-cycle* and an element of the submodule $\partial_{n+1}K_{n+1}$ an *n-boundary*. The coset of a cycle is written $\mathrm{cls}\; c = c + \partial K_{n+1}$ or as $\{c\}$. Two n-cycles in the same homology class $(\mathrm{cls}\; c = \mathrm{cls}\; c')$ are said to be *homologous* $(c \sim c')$. This is the case if they differ by a boundary.

A *subcomplex* S of K is a family of submodules $S_n \subseteq K_n$, such that $\partial S_n \subseteq S_{n+1}$. Then S is itself a complex with induced boundary. K/S is in this case a *quotient complex*.

A complex K is *positive* if $K_n = 0$, $n < 0$, while it is *negative* if $K_n = 0$, $n > 0$.

If K and K' are two complexes, a *chain transformation* $f : K \to K'$ is a family of module homomorphisms $f_n : K_n \to K'_n$ such that $\partial'_n f_n = f_{n-1}\partial_n$. The function $H_n(f) = f_*$ defined by $f_*(c + \partial K_{n+1}) = fc + \partial K'_{n+1}$ is a homomorphism $H_n(f) : H_n(K) \to H_n(K')$. H_n is a covariant functor from $\mathcal{C}$ (chain complexes and chain transformations) to $\mathcal{C}$ (modules). It is termed the *homology* functor.

A *chain homotopy* s between two chain transformations $f , g : K \to K'$ is a family of module homomorphisms $s_n : K_n \to K'_{n+1}$ such that

$$\partial'_{n+1}s_n + s_{n-1}\partial_n = f_n - g_n \quad .$$

We write $s : f \simeq g$. Algebraically we have

Theorem 1.1. If $s : f \simeq g : K \to K'$ then

$$H_n(f) = H_n(g) : H_n(K) \to H_n(K') .$$

A chain transformation $f : K \to K'$ is said to be a *chain equivalence* if there is another chain transformation $g : K' \to K$ and homotopies $s : gf \simeq 1_K$ and $t : fg \simeq 1_{K'}$. Since $H_n(1_K) = 1_{H_n(K)}$ we have

Corollary 1.2. If $f : K \to K'$ is a chain equivalence, then the induced map $H_n(f) : H_n(K) \to H_n(K')$ is an isomorphism for all n.

Proposition 1.3. Chain homotopies $s : f \simeq g : K \to K'$ and $s' : f' \simeq g' : K' \to K''$ yield a composite chain homotopy

$$f's + s'g : f'f \simeq g'g : K \to K'' .$$

Each module A may be regarded as a "trivial" positive complex with $A_0 = A$ and $A_n = 0$ for $n \neq 0$. A *complex* (K,e) *over* A is a positive complex K together with a chain transformation $e : K \to A$, such that e is simply a module homomorphism $e : K_0 \to A$.

2. Homology Groups of a Complex over a Module

Let $K = \{K_n, \partial_n\}$ be a chain complex of right R-modules and let ${}_RA$ and G_R be arbitrary R-modules. We can form two new complexes of abelian groups

$$K \otimes A : \ldots \to K_{n+1} \otimes A \xrightarrow{\partial_{*n+1}} K_n \otimes A \xrightarrow{\partial_{*n}} K_{n-1} \otimes A \to \ldots$$

$$\mathrm{Hom}\ (K,G) : \ldots \to \mathrm{Hom}\ (K_{n-1},G) \xrightarrow{\delta^{n-1}} \mathrm{Hom}\ (K_n,G) \xrightarrow{\delta^{n}} \mathrm{Hom}\ (K_{n+1},G) \to \ldots$$

where $\partial_{*n} = \partial_n \otimes 1$ and $\delta^n = (-1)^{n+1} \partial^*_{n+1}$ for $\partial^*_{n+1} = \mathrm{Hom}\ (\partial_{n+1},1)$. The n-dimensional *homology* group $H_n(K;A)$ of K over A is defined as

$$H_n(K;A) = H_n(K \otimes A) = \mathrm{Ker}\ \partial_{*n}/\mathrm{Im}\ \partial_{*n+1}$$

where the elements of $K_n \otimes A$, $\mathrm{Ker}\ \partial_{*n}$ and $\mathrm{Im}\ \partial_{*n+1}$ are the n-chains, n-cycles and n-boundaries, respectively. The n-dimensional *cohomology* group $H^n(K;G)$ of K over G is defined as

$$H^n(K;G) = H^n(\mathrm{Hom}\ (K,G)) = \mathrm{Ker}\ \delta^n/\mathrm{Im}\ \delta^{n-1} .$$

The elements of $\mathrm{Hom}\ (K_n,G)$, $\mathrm{Ker}\ \partial^n$ and $\mathrm{Im}\ \partial^{n-1}$ are termed the *n-cochains*, *n-cocycles* and *n-coboundaries*, respectively.

In the case $A = R = G$, we have $K \otimes A \cong K$ and $\mathrm{Hom}\ (K,G) \cong K$. Then $H_n(K;A) = H_n(K)$ and $H^n(K;G) = H^n(K)$, the homology and cohomology groups, respectively, of the complex K.

$H_n(K;A)$ is a covariant bifunctor while $H^n(K;G)$ is a functor contravariant in the first variable and covariant in the second, both from $\mathcal{C}$ (chain complexes) $\times$ $\mathcal{C}$ (modules) to $\mathcal{C}$ (abelian groups).

3. The Exact Homology Sequence

Consider any short exact sequence

$$E : 0 \to K \xrightarrow{\kappa} L \xrightarrow{\sigma} M \to 0 \tag{3.1}$$

of chain complexes and chain transformations κ , σ . The first transformation κ has kernel zero, but the induced map $H_n(\kappa) : H_n(K) \to H_n(L)$ may have non trivial kernel. However, we are able to define a *connecting homomorphism*

$$\partial_E : H_{n+1}(M) \to H_n(K)$$

such that $\operatorname{Im} \partial_E = \operatorname{Ker} H_n(\kappa)$. In fact ∂_E is defined for any cycle $m \in M_{n+1}$ by

$$\partial_E(\operatorname{cls} m) = \operatorname{cls} c \text{ , where } \kappa c = \partial \ell \text{ , } \sigma \ell = m \text{ , for some } \ell \text{ .}$$

c is usually denoted by $\kappa^{-1}\partial\sigma^{-1}(m)$. This gives

Theorem 3.1. (The Exact Homology Sequence) For each short exact sequence (3.1) of chain complexes the corresponding long sequence

$$\cdots \to H_{n+1}(M) \xrightarrow{\partial_E} H_n(K) \xrightarrow{\kappa_*} H_n(L) \xrightarrow{\sigma_*} H_n(M) \xrightarrow{\partial_E} H_{n-1}(K) \to \cdots$$

of homology groups, with maps the connecting homomorphism

∂_E , $\kappa_* = H_n(\kappa)$ and $\sigma_* = H_n(\sigma)$ is exact.

Consider the category $\mathcal{E}$ of short exact sequences and chain transformations. A morphism $E \to E'$ in $\mathcal{E}$ is a triple (f,g,h) of chain transformations making the diagram

$$\begin{array}{ccccccccc} E: 0 & \to & K & \to & L & \to & M & \to & 0 \\ & & f\downarrow & & g\downarrow & & h\downarrow & & \\ E': 0 & \to & K' & \to & L' & \to & M' & \to & 0 \end{array}$$

commutative. It follows that for each $E \in \mathcal{E}$, the connecting homomorphism $\partial_E : H_{n+1}(M) \to H_n(K)$ is natural.

The corresponding exact sequence on cohomology is given by

Theorem 3.2. If G is an R-module and E a short exact sequence (3.1) of complexes of R-modules which splits as a sequence of modules, then there is for each dimension n , a natural connecting homomorphism $\partial_E : H^n(K;G) \to H^{n+1}(M;G)$ such that the sequence of cohomology groups

$$\ldots \to H^n(M;G) \xrightarrow{\sigma^*} H^n(L;G) \xrightarrow{\kappa^*} H^n(K;G) \xrightarrow{\partial_E} H^{n+1}(M;G) \quad \ldots$$

is exact.

If $0 \to G' \to G \to G'' \to 0$ is a short exact sequence of modules and K a complex consisting of projective modules, then there exists another connecting homomorphism $H^n(K;G'') \to H^{n+1}(K;G)$ yielding the long exact sequence

$$\ldots \to H^n(K;G') \to H^n(K;G) \to H^n(K;G'') \to H^{n+1}(K;G) \to \ldots \qquad (3.2)$$

of cohomology groups.

4. Singular Homology

Homology and cohomology theories in topological categories are defined on what S.T. Hu in [5] calls "admissable categories", all such categories being subcategories of the category of topological pairs. As an example of the use of complexes we give a brief description of the singular homology theory of a topological space X or a pair (X, ϕ).

For any topological space X, a *singular n-simplex* T in X is a continuous map $T : \Delta^n \to X$, where Δ^n is the subspace of R^{n+1} consisting of all points $(x_o, x_1, \ldots, x_n)$ of R^{n+1} satisfying

$$\sum_{i=o}^{n} x_i = 1 \quad , \qquad x_i \geq 0 \quad i = 0, 1, \ldots, n \quad .$$

Let $S_n(X) = \text{Map}(\Delta^n, X)$ denote the set of all singular n-simplicies in X. For $n > 0$, let $T : \Delta^n \to X$ denote an arbitrary singular n-simplex in X. For every $t = 0, 1, \ldots, n$, the composition $t\, e_n^i$ of the maps $T : \Delta^n \to X$ and $e_n^i : \Delta^{n-1} \to \Delta^n$

$$\Delta^{n-1} \to \Delta^n \to X$$

with $e_n^i(x_o, x_1, \ldots, x_{n-1}) = (x_o, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{n-1})$ is a singular

(n - 1)-simplex in X called the i^{th} *face* of T. For every integer $n > 0$, and every $i = 0, 1, \ldots, n$, the assignment $T \to T e^{i}$ defines a function

$$\sigma_i : S_n(X) \to S_{n-1}(X)$$

called the i^{th} *face operation* on $S_n(X)$.

Let $C_n(X)$ be the free abelian group generated by the set $S_n(X)$. The i^{th} face operation on $S_n(X)$ defines a homomorphism

$$\partial : C_n(X) \to C_{n-1}(X)$$

given by

$$\partial(T) = \sum_{i=o}^{n} (-1)^i \sigma_i(T) \qquad T \varepsilon S_n(X) \quad .$$

This is referred to as the boundary operator. It is easily seen that $\partial\partial = 0$.

$C(X) = \{C_n(X), \partial\}$ is a complex called the *singular complex* of the space X. The n-dimensional *singular homology* group $H_n(X)$ of the space X is defined to be the n^{th} homology group $H_n(C(X))$ of the singular complex.

Let X denote an arbitrary topological space and G any abelian group. Consider the singular complex $C(X)$ of abelian groups. We have induced two new complexes

$$C(X) \otimes G = \{C_n(X) \otimes G \, , \, \partial_*\}$$

$$\mathrm{Hom}\,(C(X)\,,\,G) = \{\mathrm{Hom}\,(C_n(X)\,,\,G)\,,\,\delta\} \quad .$$

The n-dimensional *singular homology* group $H_n(X;G)$ of X over G is the n^{th} homology group of the complex $C(X) \otimes G$, while the n-dimensional *singular cohomology* group $H^n(X;G)$ of X over G is the n^{th} homology group of the complex $\mathrm{Hom}\,(C(X)\,,\,G)$. In particular, if G is the group Z of all integers, we have $H_n(X;G) = H_n(X)$ and $H^n(X;G) = H^n(X)$, the homology and cohomology groups, respectively, of the space X.

CHAPTER III

EXTENSIONS AND COHOMOLOGY

This chapter begins with the definition of Ext^n which is immediately shown to be a group. Ext^1 is then used (§ 3) to calculate the cohomology of a complex of free abelian groups over a group from the homology of the complex. $\mathrm{Ext}^n(C,A)$ is shown to be exactly $H^n(X;A)$ for X an arbitrary projective resolution of C. The chapter ends with an axiomatic description of Ext .

1. Extensions

Let A and C be modules over a fixed ring R . An *n-fold extension* of A by C is a long exact sequence

$$0 \to A \to B_{n-1} \to \dots \to B_o \to C \to 0 \qquad (1.1)$$

of R-modules and R-module homomorphisms. If S and S' are n-fold extensions, a *morphism* $\Gamma : S \to S'$ is a family of homomorphisms $(\alpha, \dots, \gamma)$ forming a commutative diagram

$$\begin{array}{ccccccccccccc}
S: & 0 & \to & A & \to & B_{n-1} & \to & \dots & \to & B_o & \to & C & \to & 0 \\
\Gamma \downarrow & & & \alpha \downarrow & & \downarrow & & & & \downarrow & & \gamma \downarrow & & \\
S': & 0 & \to & A' & \to & B'_{n-1} & \to & \dots & \to & B'_o & \to & C' & \to & 0
\end{array} .$$

The morphism Γ is said to have *fixed ends* if $A = A'$, $C = C'$ and α and γ are identities.

Two n-fold extensions S , S' are *congruent* $(S \equiv S')$ if there is a chain $S = S_o, S_1, \ldots, S_{k-1}, S_k = S$ of exact sequences such that for each i $(0 \leq i \leq k - 1)$ there is a morphism $S_i \to S_{i+1}$ or $S_{i+1} \to S_i$ which has fixed ends. Congruence of extensions is an equivalence relation. We let $\mathrm{Ext}^n(C,A)$ stand for the set of equivalence classes $\sigma = \mathrm{cls}\ S$ of n-fold extensions S of the R-module A by the R-module C . Write $S \varepsilon \varepsilon \mathrm{Ext}^n(C,A)$ for $S \varepsilon \sigma \varepsilon \mathrm{Ext}^n(C,A)$.

Let $S \varepsilon \sigma \varepsilon \mathrm{Ext}^n(D,C)$, $S' \varepsilon \tau \varepsilon \mathrm{Ext}^m(C,A)$. The *Yoneda Composite* $\tau\sigma \varepsilon \mathrm{Ext}^{m+n}(D,C)$ is defined by composition $S'S$ of representatives of τ and σ as follows. It

$$S : 0 \to D \to B_{n-1} \to \ldots \to B_o \xrightarrow{e} C \to 0 \qquad S' : 0 \to C \xrightarrow{\mu} B'_{m-1} \to \ldots \to B'_o \to A \to 0$$

then SS' is defined as

$$SS' : 0 \to D \to B_{n-1} \to \ldots \to B_o \xrightarrow{\mu e} B'_{m-1} \to \ldots \to B'_o \to A \to 0$$

which is seen to be an $(m + n)$-fold sequence. $\tau\sigma = \mathrm{cls}\ SS'$ is well defined, and the composition is associative.

The n-fold exact sequence (1.1) may be written as a product $S = E_n E_{n-1} \ldots E_1$ of short exact sequences by putting

$$E_i : 0 \to K_i \to B_i \to K_{i-1} \to 0$$

where $K_i = \text{Ker}\ [B_i \to B_{i-1}]$.

A 1-fold extension is termed simply an *extension* and Ext is written for Ext^1 .

Lemma 1.1. If $E \in \text{Ext}\ (C,A)$ for R-modules C and A and $\gamma : C' \to C$ is a module homomorphism, there exists an extension $E' \in \text{Ext}\ (C',A)$ and a morphism $\Gamma = (1_A,\beta,\gamma) : E \to E'$. The pair (Γ,E') is unique up to a congruence of E' .

Proof. Existence. In the diagram

$$\begin{tikzcd} E' : 0 \arrow[r] & A \arrow[r, dashed, "\kappa'"] \arrow[d, no head] & ? \arrow[r, dashed, "\sigma'"] \arrow[d, dashed, "\beta"] & C' \arrow[r] \arrow[d, "\gamma"] & 0 \\ E : 0 \arrow[r] & A \arrow[r, "\kappa"] & B \arrow[r, "\sigma"] & C \arrow[r] & 0 \end{tikzcd}$$

replace ? by the module $B' = \{(b,c') \in B \oplus C',\ \sigma b = \gamma c'\}$. Define the required maps κ' , σ' and β by $\kappa' a = (\kappa a,0)$, $\sigma'(b,c') = c'$ and $\beta(b,c') = b$, respectively. Then $E' = (\kappa',\sigma') : A \rightarrowtail B' \twoheadrightarrow C'$ is an extension of A by C' and $\Gamma = (1_A,\beta,\gamma)$ is the required morphism.

Uniqueness. Take any other E'' with a morphism $\Gamma'' = (1_A,\beta'',\gamma) : E'' \to E$. If B'' is the middle module in E'' , define $\beta' : B'' \to B'$ by $\beta' b'' = (\beta'' b'',\ \sigma'' b'')$; then $\Gamma_o = (1_A,\beta',1_{C'})$:

$E'' \to E'$ is a congruence and the composite $E'' \to E' \to E$ is Γ'', so that the diagram $\Gamma : E' \to E$ is unique up to a congruence Γ_o of E'.

We call $E' = E\gamma$ the *composite* of the extension E and the homomorphism γ.

Lemma 1.2. Under the hypothesis of the previous lemma each morphism $\Gamma_1 = (\alpha_1, \beta_1, \gamma_1) : E_1 \to E$ of extensions with $\gamma_1 = \gamma$ can be factored through $\Gamma : E\gamma \to E$.

Proof. We need a morphism $E_1 \to E\gamma$ making the diagram

$$\begin{tikzcd} & E\gamma \arrow[d, "{(1,\beta,\gamma)}"] \\ E_1 \arrow[ur, dashed] \arrow[r, "{(\alpha_1,\beta_1,\gamma)}"] & E \end{tikzcd}$$

commutative. $E_1 = (\kappa_1, \sigma_1) : A_1 \rightarrowtail B_1 \twoheadrightarrow C'$. Define $\beta' : B_1 \to B'$ by $\beta' b_1 = (\beta_1 b_1, \sigma_1 b_1)$. Then the morphism $(\alpha_1, \beta', 1)$ satisfies the requirements.

We have two lemmas dual to the two given above.

Lemma 1.3. For $E \in \text{Ext}(C,A)$ and $\alpha : A \to A'$ there exists an $E' \in \text{Ext}(C,A')$ and a morphism $\Gamma = (\alpha, \beta, 1_C) : E \to E'$. The pair (Γ, E') is unique up to a congruence of E'.

Proof. The diagram

$$\begin{tikzcd} E : 0 \arrow[r] & A \arrow[r, "\kappa"] \arrow[d, "\alpha"'] & B \arrow[r, "\sigma"] \arrow[d, dashed, "\beta"'] & C \arrow[r] \arrow[d, no head] & 0 \\ E' : 0 \arrow[r] & A' \arrow[r, dashed, "\kappa'"] & ? \arrow[r, dashed, "\sigma'"] & C \arrow[r] & 0 \end{tikzcd}$$

is completed to commutativity by defining $B' = (A' \oplus B)/N$ where $N = \{(-\alpha a, \kappa a) \; ; \; a \in A\}$, $\kappa' a' = (a',0) + N$, $\beta b = (0,b) + N$ and $\sigma'((a', b) + N) = \sigma b$. The proof of uniqueness is similar to the proof in Lemma 1.2 . αE is written for E' .

Lemma 1.4. Under the hypothesis of the previous lemma, any morphism $\Gamma_1 = (\alpha_1,\beta_1,\gamma_1) : E \to E_1$ with $\alpha_1 = \alpha$ can be factored through $E \to \alpha E$.

Proof. The diagram

$$\begin{tikzcd} \alpha E \arrow[dr, dashed] & \\ E \arrow[u, "{(\alpha,3,1)}"] \arrow[r, "{(\alpha_1,\beta_1,\gamma_1)}"] & E_1 \end{tikzcd}$$

is completed to commutativity by defining a morphism $(1_{A'},\beta',\gamma_1)$: $\alpha E \to E_1$ where $\beta' : (A' \oplus B)/N \to B_1$ is given by $\beta'((a',b) + N) = \kappa_1 a' + b_1 b$.

Lemma 1.1 shows $E' = E\gamma$ unique, hence implies the congruences

$$E1_C \equiv E \quad \text{and} \quad E(\gamma\gamma') \equiv (E\gamma)\,\gamma' ,$$

showing Ext (C,A) to be contravariant functor of C . Lemma 1.3 shows Ext (C,A) to be a covariant functor in A . The following lemma verifies that Ext is actually a bifunctor.

Lemma 1.5. For α , γ , and E as in Lemmas 1.1 and 1.3 there is a congruence of extensions $\alpha(E\gamma) \equiv (\alpha E)\gamma$.

Proof. We have

$$E\gamma \xrightarrow{(1,\beta_1,\gamma)} E \xrightarrow{(\alpha,\beta_2,1)} \alpha E$$

with composition $(\alpha, \beta_2\beta_1,\gamma) : E\gamma \to \alpha E$. Lemma 1.2 gives a factorization of this morphism by

$$E\gamma \xrightarrow{(\alpha,\beta',1)} (\alpha E)\gamma \xrightarrow{(1,\beta\ ,\gamma)} \alpha E \ .$$

The uniqueness up to a congruence of E' in Lemma 1.3 gives $\alpha(E\gamma) \equiv (\alpha E)\gamma$.

These lemmas are used to prove:

Proposition 1.6. For any extension $E = (\kappa,\sigma)$, the composite extensions κE and $E\sigma$ are split.

Lemma 1.3 implies that κE splits while the splitting of $E\sigma$ follows from 1.1 .

Proposition 1.7. Any morphism $\Gamma_1 = (\alpha,\beta,\gamma) : E \to E'$ of extensions

implies the congruence $\alpha E \equiv E'\gamma$.

Proof. By Lemma 1.4 the morphism Γ_1 can be factored through $\Gamma : E \to \alpha E$ as $\Gamma_1 = \Gamma_2\Gamma$ where $\Gamma_2 = (1_{A'}, \beta', \gamma) : \alpha E \to E'$. Γ_2 characterizes αE as $E'\gamma$ by Lemma 1.1 .

If $S \in \varepsilon\, \mathrm{Ext}^n (C,A)$, $\alpha \in \mathrm{Hom}\,(A',A)$, $\gamma \in \mathrm{Hom}\,(C,C')$ we define αS and $S\gamma$ for $S = E_n E_{n-1} \dots E_1$ by

$$\alpha S = (\alpha E_n)\, E_{n-1} \dots E_1$$

$$\text{and} \quad S\gamma = E_n \dots E_2(E_1\gamma) \quad .$$

Lemma 1.8. If $S \in \varepsilon\, \mathrm{Ext}^n (C,A)$, $S' \in \varepsilon\, \mathrm{Ext}^m(D,C')$ and $\gamma \in \mathrm{Hom}\,(C,C')$, then $(S\gamma)S' \equiv S(\gamma S')$.

Proof. $(S\gamma)S' = E_n \dots (E_1\gamma)\, E'_m \dots E'_1$, so it suffices to take short exact sequences

$$E_1 : K_o \rightarrowtail B_o \twoheadrightarrow C \quad \text{and} \quad E'_m : C \rightarrowtail B'_{m-1} \twoheadrightarrow K'_m \ .$$

Form $E_1\gamma$ and $\gamma E'_m$. Joining the commutative diagrams

$$\begin{array}{ccccccc}
E_1\gamma : & K_o & \rightarrowtail & K_o & \twoheadrightarrow & C' & \\
& \| & & \downarrow & & \downarrow \gamma & \\
E_1 : & K_o & \rightarrowtail & B_o & \twoheadrightarrow & C &
\end{array}
\qquad
\begin{array}{ccccccc}
C' & \rightarrowtail & B'_{m-1} & \twoheadrightarrow & K'_m & : & E'_m \\
\gamma \downarrow & & \downarrow & & \| & & \\
C & \rightarrowtail & L & \twoheadrightarrow & K'_m & : & \gamma E'_m
\end{array}$$

along $\gamma : C' \to C$ yields a morphism $(E_1\gamma)\, E'_m \to E_1(\gamma E'_m)$.

Proposition 1.9. If $S = E_n \dots E_1 \equiv S' = E'_n \dots E'_1$ in $\mathrm{Ext}^n(C,A)$, then S can be transformed into S' by a finite number of switches between sequences of type $\dots (E_i\beta)\, E_{i+1} \dots$ and $\dots E_i(\beta E_{i+1}) \dots$, for β a suitable homomorphism.

Proof. Suppose we have

$$\begin{tikzcd}
S : 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B_{n-1} \arrow[r] \arrow[d] & \dots \arrow[r] & B_o \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
S' : 0 \arrow[r] & A \arrow[r] & B'_{n-1} \arrow[r] & \dots \arrow[r] & B'_o \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

Then the short exact sequences of the decomposition are related by the morphisms $E_i \to E'_i$

$$\begin{tikzcd}
E_i : 0 \arrow[r] & K_i \arrow[r] \arrow[d, "\beta_i"'] & B_i \arrow[r] \arrow[d] & K_{i-1} \arrow[r] \arrow[d, "\beta_{i-1}"'] & 0 \\
E'_i : 0 \arrow[r] & K'_i \arrow[r] & B'_i \arrow[r] & K'_{i-1} \arrow[r] & 0
\end{tikzcd}$$

where $\beta_n = 1_A$ and $\beta_o = 1_C$. Prop. 1.7 gives $\beta_i E_i \equiv E'_i \beta_{i-1}$.
Hence by Lemma 1.8

$$\begin{aligned}
S = (\beta_n E_n)\, E_{n-1} \dots E_1 &\equiv (E'_n \beta_{n-1})\, E_{n-1} \dots E_1 \\
&\equiv E'_n(\beta_{n-1} E_{n-1}) \dots E_1 \\
&\equiv \dots \\
&\equiv E'_n E'_{n-1} \dots (E'_1 \beta_o) = S' .
\end{aligned}$$

This proposition gives

Proposition 1.10. Each morphism $\Gamma = (\alpha,\dots,\gamma) : S \to S'$ of n-fold

extensions yields a congruence $\alpha S \equiv S'\gamma$.

2. The Group Ext

The *diagonal and codiagonal* maps of a module C

$$\Delta_C : C \to C \oplus C \quad \text{and} \quad \nabla_C : C \oplus C \to C$$

are given by $\Delta_C(c) = (c,c)$ and $\nabla_C(c,c') = c + c'$. The usual sum $f + g$ of two homomorphisms $f,g : C \to A$ can be written as

$$f + g = \nabla_A(f \oplus g)\, \Delta_C \ .$$

Two n-fold extensions $S \in \sigma \in \mathrm{Ext}^n(C,A)$, $S' \in \tau \in \mathrm{Ext}^n(C,A)$ have a *direct sum* $S \oplus S' \in \in \mathrm{Ext}^n(C \oplus C', A \oplus A')$ found by taking direct sums of corresponding modules and maps in S and S' . The congruence class of $S \oplus S'$ depends only on the classes σ and τ and hence may be written $\sigma \oplus \tau$.

Finally the *Baer sum* is defined for $\sigma , \tau \in \mathrm{Ext}^n(C,A)$ by the formula

$$\sigma + \tau = \nabla_A(\sigma \oplus \tau)\, \Delta_C \ .$$

We state two lemmas whose proofs are straightforward.

Lemma 2.1. There is a natural isomorphism $\omega : A' \oplus A \to A \oplus A'$

given by $\omega(a',a) = (a,a')$ and having the properties

$$\omega(S \oplus S') \equiv (S' \oplus S)\omega \quad ,$$

$$\nabla\omega = \nabla \quad \text{and} \quad \omega\Delta = \Delta \quad .$$

Lemma 2.2. For any extension S we have

$$\Delta S \equiv (S + S)\Delta \quad \text{and}$$

$$S\nabla \equiv \nabla(S + S) \quad .$$

Lemma 2.3. For S , $S' \varepsilon \operatorname{Ext}^n (C,A)$, $T \varepsilon \operatorname{Ext}^m (D,C)$, and α , $\alpha' \varepsilon \operatorname{Hom} (A',A)$ we have

i. $(S + S')T \equiv ST + S'T$

ii. $(\alpha + \alpha')S \equiv \alpha S + \alpha' S$

iii. $\alpha(S + S') \equiv \alpha S + \alpha S'$

and dually.

Proof. The proofs are straightforward and just manipulative. We prove (i) only.

$$\begin{aligned}
(S + S')T &= (\nabla(S \oplus S')\Delta)T \\
&= \nabla((E_n \oplus E_n') \ldots (E_1 \oplus E_1'))\, \Delta\, (F_m \ldots F_1) \\
&\equiv \nabla(E_n \oplus E_n') \ldots (E_1 \oplus E_1')(\Delta F_m) \ldots F_1 \\
&\equiv \nabla(E_n \oplus E_n') \ldots (E_1 \oplus E_1')(F_m \oplus F_m)\Delta \ldots F_1 \\
&\equiv \ldots
\end{aligned}$$

$$\equiv \nabla(E_n \oplus E'_n) \ldots (E_1 \oplus E'_1)(F_m \oplus F'_m) \ldots (F_1 \oplus F_1) \Delta$$
$$\equiv \nabla(ST \oplus S'T) \Delta$$
$$= ST + S'T \, .$$

Theorem 2.4. For R-modules A and C, the set of congruence classes of n-fold extensions of A by C, $\mathrm{Ext}^n(C,A)$, under addition given by the Baer sum is an abelian group.

Proof. Associativity

$$S + (S' + S'') = \nabla(S \oplus (\nabla(S' \oplus S'') \Delta) \Delta$$
$$= \nabla(1 \oplus \nabla)(S \oplus (S' \oplus S''))(1 \oplus \Delta) \Delta \quad .$$

Similarly $(S + S') \oplus S'' = \nabla(\nabla \oplus 1)((S \oplus S') \oplus S'')(\Delta \oplus 1) \Delta$. The identification of $S \oplus (S \oplus S)$ with $(S \oplus S) \oplus S$ implies that $\nabla(1 \oplus \nabla)$ and $\nabla(\nabla \oplus 1)$, $(1 \oplus \nabla) \nabla$ and $(\nabla \oplus 1) \nabla$ are the same.

Commutativity. $S + S' = \nabla(S \oplus S')\Delta = \nabla\omega(S \oplus S') \Delta \equiv \nabla(S' \oplus S) \omega\Delta = \nabla(S' \oplus S) \Delta = S' + S$.

$S_o : 0 \to A \xrightarrow{1} A \to 0 \to \ldots \to 0 \to C \xrightarrow{1} C \to 0$ represents the zero element of $\mathrm{Ext}^n(C,A)$.

To obtain the inverse of $S \,\varepsilon\, \varepsilon\, \mathrm{Ext}^n(C,A)$ we note that $0_A S$ is of the form $0 \to A \to A \oplus B' \to B_{n-2} \to \ldots \to C \to 0$, and that there is a morphism $0_A S \to S_o$ with fixed ends. By Prop. 2.3 (ii) ,

$S_o \equiv 0_A S = (1_A + (-1_A))\, S \equiv 1_A S + (-1_A)\, S = S + (-1_A)\, S$.

Theorem 2.5. $\operatorname{Ext}^n(\cdot\,,\cdot)$ is an additive bifunctor, contravariant in the first and covariant in the second variable from $\mathcal{C}$ (modules) $\times$ $\mathcal{C}$ (modules) to $\mathcal{C}$ (abelian groups) .

Proof. For homomorphisms $\alpha : A \to A'$, $\gamma : C \to C'$ there is a map $\operatorname{Ext}^n(\gamma,\alpha) : \operatorname{Ext}^n(C,A') \to \operatorname{Ext}^n(C',A)$ given by $\operatorname{Ext}^n(\gamma,\alpha)\, S = (\alpha S)\,\gamma \equiv \alpha(S\gamma)$. This is a group homomorphism by Lemma 2.3 (iii) and its dual. The additive property follows from Lemma 2.3 (ii) and its dual.

The additive property of the bifunctor gives

$$\operatorname{Ext}^n(A \oplus B,G) \cong \operatorname{Ext}^n(A,G) \oplus \operatorname{Ext}^n(B,G)$$
$$\text{and}\quad \operatorname{Ext}^n(A,G \oplus H) \cong \operatorname{Ext}^n(A,G) \oplus \operatorname{Ext}^n(A,H) \quad .$$

Furthermore, any short extension by a projective module splits. Hence for P projective

$$\operatorname{Ext}^n(P,G) = 0 \qquad n > 0 \quad .$$

Similarly for Q injective

$$\operatorname{Ext}^n(A,Q) = 0 \qquad n > 0 \quad .$$

3. The University Coefficient Theorem for Cohomology

As an application of the functor Ext^1 we give a method of "calculating" the cohomology groups of complex (of free abelian groups) over a group from the homology of that complex.

Theorem 3.1. Let K be a complex of free abelian groups K_n and G an arbitrary abelian group. Then for each dimension n there is an exact sequence

$$0 \to \mathrm{Ext}\,(H_{n-1}(K),G) \overset{\beta}{\to} H^n(K;G) \overset{\alpha}{\to} \mathrm{Hom}\,(H_n(K),G) \to 0$$

with homomorphisms β and α natural in K and G. The sequence splits by a homomorphism which is natural in G but not in K.

For a detailed proof we refer the reader to MacLane [8] pp. 77-88. The second map α is defined on a cohomology class, cls f, as follows. Each n-cocycle of $\mathrm{Hom}\,(K,G)$ is a homomorphism $f : K_n \to G$ which vanishes on ∂K_{n+1}, so induces $f_* : H_n(K) \to G$. If $f = \delta g$ is a coboundary, it vanishes on cycles, so $(\delta g)_* = 0$. Define $\alpha(\mathrm{cls}\, f) = f_*$. Then α is an epimorphism with kernel $\mathrm{Ext}\,(H_{n-1}(K),G)$.

Corollary 3.2. If $f : K \to K'$ is a chain transformation between complexes K and K' of free abelian groups with $f_* : H_n(K) \cong H_n(K')$ for all n, then for any group G, $f^* : H^n(K';G) \to H^n(K;G)$ is an isomorphism.

Proof. Since the maps α and β are natural in K , the diagram

$$\begin{array}{ccccc} \operatorname{Ext}(H_{n-1}(K'),G) & \rightarrowtail & H^n(K';G) & \twoheadrightarrow & \operatorname{Hom}(H_n(K'),G) \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Ext}(H_{n-1}(K),G) & \rightarrowtail & H^n(K;G) & \twoheadrightarrow & \operatorname{Hom}(H_n(K),G) \end{array}$$

is commutative. The isomorphisms $f_* : H_n(K) \cong H_n(K')$ make the outside vertical maps $\operatorname{Ext}(f_*,1_G)$ and $\operatorname{Hom}(f_*,1_G)$ isomorphisms. By the short five lamma the middle map is also an isomorphism.

4. Resolutions

We defined in I §1 a complex (X,e) over an R-module C as a sequence

$$\ldots \to X_n \overset{\partial}{\to} X_{n-1} \to \ldots \to X_1 \overset{\partial}{\to} X_o \overset{e}{\to} C \to 0 \qquad (4.1)$$

of R-modules X_n and module homomorphisms ∂ such that the composition of any two is zero and also $e\partial = 0$. A *resolution* of C is an exact sequence (4.1) ; that is, a complex (X,e) over C with homology $H_n(X) = 0$ for $n \neq 0$ and $H_o(X) \cong C$. The complex X if *free* or *projective* according as each X_n is free or projective.

Any module C is a quotient F_o/R_o of some free (projective) module F_o . The submodule R_o is again a quotient F_1/R_1 of a suitable free (projective) module F_1 . Continuing this process yields a free (projective) resolution

$$\ldots \to F_n \to \ldots \to F_1 \to F_o \to C \to 0$$

of C .

By a *complex* (e,Y) *under* the module A we mean a sequence

$$0 \to A \xrightarrow{e} Y^o \to Y' \to \ldots \to Y^n \to \ldots$$

such that the composition of any two successive homomorphisms is zero. If the sequence is exact it is termed a *coresolution* of A ; if each Y^n is injective, (e,Y) is an *injective complex under* A . Since every module is a submodule of an injective module, every module has an injective coresolution.

Theorem 4.1. (Comparison Theorem for Projectives)
If $\gamma : C \to C'$ is a homomorphism of modules, while (X,e) is a projective complex over C' , then there is a chain transformation $f : X \to X'$ with $e'f = \gamma e$. Any two such chain transformations are chain homotopic.

Proof. Since X_o is projective and e' an epimorphism, $\gamma e : X_o \to C'$ can be lifted to $f_o : X_o \to X'_o$ with $e'f_o = \gamma e$. By induction it suffices to construct f_n , given f_o , ..., f_{n-1} such that the diagram

$$\begin{array}{ccccccccccccc}
X_n & \xrightarrow{\partial_n} & X_{n-1} & \xrightarrow{\partial_{n-1}} & X_{n-2} & \to & \cdots & \to & X_o & \xrightarrow{e} & C & \to & 0 \\
\downarrow f_n & & \downarrow f_{n-1} & & \downarrow f_{n-2} & & & & \downarrow f_o & & \downarrow \gamma & & \\
X'_n & \xrightarrow{\partial'_n} & X'_{n-1} & \xrightarrow{\partial'_{n-1}} & X'_{n-2} & \to & \cdots & \to & X'_o & \xrightarrow{e'} & C & \to & 0
\end{array}$$

is commutative. By commutativity $\partial'_{n-1} f_{n-1}\partial_n = f_{n-2}\partial_{n-1}\partial_n = 0$. Hence $\operatorname{Im} f_{n-1}\partial_n \subseteq \operatorname{Ker} \partial'_{n-1}$. X_n projective implies that $f_{n-1}\partial_n$ can be lifted to $f_n : X_n \to X'_n$ with $\partial' f_n = f_{n-1}\partial_n$. The construction of the homotopy is similar.

The dual theorem for injectives is now stated, without proof.

Theorem 4.2. (Comparison Theorem for Injectives) If $\alpha : A \to A'$ is a module homomorphism, (e,Y) a coresolution under A and (e',Y') an injective complex under A', then there is a chain transformation $f : Y \to Y'$ with $e'\alpha = fe$. Any two such chain transformations are homotopic.

Lemma 4.3. Under the hypothesis of Thm. 4.1, let $f : X \to X'$ be a chain transformation lifting $\gamma : C \to C'$. Suppose that there is a homomorphism $t : C \to X'_o$ with $e't = \gamma$. Then there exists homomorphisms $s_n : X_n \to X'_{n+1}$ such that for all $n = 0, 1, \ldots$

$$\partial' s_o + te = f_o \ , \quad \partial' s_{n+1} + s_n\partial = f_{n+1}$$

The proof is straightforward using induction and the projectivity of the X_n's.

Let A be a fixed module; applying the functor $\mathrm{Hom}\,(\cdot,A)$ to a resolution (4.1) gives a complex $\mathrm{Hom}\,(X,A)$ which may have non-trivial homology

$$H^n(X;A) = H^n(\mathrm{Hom}\,(X,A)) \quad .$$

However we have

Corollary 4.4. If X and X' are two projective resolutions of C while A is any module, then $H^n(X;A) \cong H^n(X';A)$ depends only on A and C .

Proof. By the first part of Thm. 4.1 there are chain transformations $f : X \to X'$ and $g : X' \to X$ lifting $1_C'$. By the second part of Thm. 4.1 $gf \simeq 1_X : X \to X$. Hence $(gf)^* = f^* g^* \simeq 1_{\mathrm{Hom}\,(X,A)}$: $\mathrm{Hom}\,(X,A) \to \mathrm{Hom}\,(X,A)$. Thm. 1.1 of I gives

$$H_n(f^*)\, H_n(g^*) = 1_{H^n(X;A)} \quad : H^n(X;A) \to H^n(X;A)$$

Similarly

$$H_n(g^*)\, H_n(f^*) = 1_{H^n(X';A)} \quad .$$

We now show that this function $H^n(X;A)$ of A and C for X an arbitrary projective resolution of C is exactly $\mathrm{Ext}^n\,(C,A)$. For $n = 0$, it is clear that $H^o(X;A) \cong \mathrm{Hom}\,(C,A)$, which by convention

is $\mathrm{Ext}^o(C,A)$. For $n > 0$ we have

Theorem 4.5. For R-modules C and A and a projective resolution (X,e) over C, there is a group isomorphism

$$\zeta : \mathrm{Ext}^n(C,A) \cong H^n(X;A) \qquad n = 1,2,\ldots$$

natural in both A and C.

Here we have naturality in C in the sense that if $\gamma : C' \to C$ is a homomorphism, (X',e') a projective resolution of C' and $f : X' \to X$ lifts γ, then

$$\zeta'\gamma^* = f^*\zeta : \mathrm{Ext}^n(C,A) \to H^n(X';A) .$$

Proof. ζ is defined as follows. For $n > 0$, each n-fold exact sequence in $\mathrm{Ext}^n(C,A)$ may be regarded as a resolution of C, zero beyond term A of degree n, as in the diagram

$$\begin{array}{ccccccccccccc}
\cdots \to & X_{n+1} & \to & X_n & \to & X_{n-1} & \to & \cdots & \to & X_o & \to & C & \to 0 \\
 & & & \downarrow{\scriptstyle g_n} & & \downarrow & & & & \downarrow & & \| & \\
S : 0 & & \to & A & \to & B_{n-1} & \to & \cdots & \to & B_o & \to & C & \to 0
\end{array}$$

Lift 1_C to $g : X \to S$. Then $g_n : X_n \to A$ is a cocycle of X . Define $\zeta\,(\mathrm{cls.}\ S) = \mathrm{cls}\ g_n$. Then ζ is a well defined group homomorphism.

The inverse η of ζ is defined as follows. In the resolution X, factor $\partial : X_n \to X_{n-1}$ as $X_n \xrightarrow{\partial} \partial X_n \xrightarrow{\kappa} X_{n-1}$, where κ is the injection; this yields an n-fold exact sequence $S_n(C,X)$ as in

$$\begin{tikzcd}
 & & \cdots \arrow[r] & X_{n+1} \arrow[r, "\partial"] & X_n \arrow[d, "\partial'"'] \arrow[dr, "\partial"] & & & & & \\
S_n(C,X) : & 0 \arrow[rrr] & & & \partial X_n \arrow[r, "\kappa"] \arrow[d, "h"] & X_{n-1} \arrow[r] & \cdots \arrow[r] & X_o \arrow[r] & C \arrow[r] \arrow[d, equal] & 0 \\
hS_n : & 0 \arrow[rrr] & & & A \arrow[rrrr] & & & & C \arrow[r] & 0 \ .
\end{tikzcd}$$

Any n cocycle $f : X_n \to A$ vanishes on $\partial X_{n+1} = \mathrm{Ker}\ \partial'$. Hence may be written as $f_n = h\partial'$ for some $h : \partial X_n \to A$. Construct hS_n and define $\eta(\mathrm{cls}\ f_n) = \mathrm{cls}\ (hS_n)$. Again, η is a well defined group homomorphism.

We compare the two diagrams to see that $\eta = \zeta^{-1}$. $\zeta\eta(\mathrm{cls}\ f_n) = \zeta(\mathrm{cls}\ (hS_n)) = \mathrm{cls}\ f'_n$. f_n and f'_n are two chain transformations lifting 1_C and hence there is a chain homotopy s with $f_n - f'_n = s_{n-1}\partial_n = (-1)^n\ \delta_{n-1}\ s_{n-1}$. This says that $f_n - f'_n$ is a coboundary and so $\mathrm{cls}\ f_n = \mathrm{cls}\ f'_n$ and $\zeta\eta = 1$.

$\eta\zeta\ (\mathrm{cls}\ S) = \eta(\mathrm{cls}\ g_n) = \eta(\mathrm{cls}\ h_n') = \mathrm{cls}\ hS_n$. This gives a morphism $\Gamma = (h_n, g_{n-1}, \ldots, g_o, 1) : S_n \to S$ which by lemma 1.4 can be factored

$$S_n \to hS_n \to S$$

through hS_n. The morphism $hS_n \to S$ has fixed ends and so

cls S = cls hS_n ; that is $\eta\zeta = 1$.

Corollary 4.6. If $S \in \varepsilon \operatorname{Ext}^n(C,A)$ with $n > 1$, there is a $T \equiv S$ of the form $T : 0 \to A \to B_{n-1} \to \dots \to B_o \to C \to 0$ in which the modules $B_o, \dots, B_{n-2}$ are free.

Take $T = h(S_n(C,X))$ for a suitable $h : \partial X_n \to X$ and X any free resolution of C .

Corollary 4.7. For abelian groups A and C considered as Z-modules, $\operatorname{Ext}^n(C,A) = 0$, $n > 1$.

Write $C = F/R$, F free abelian. Then R is free and $0 \to R \to F \to C \to 0$ is a free resolution which vanishes (with its cohomology) in dimensions above 1 .

For each module C a negative complex Y determines a negative complex

$$\operatorname{Hom}(C,Y) : 0 \to \operatorname{Hom}(C,Y^o) \to \operatorname{Hom}(C,Y^2) \to \dots \to \operatorname{Hom}(C,Y^n) \to \dots$$

its homology gives Ext as follows, without proof.

Theorem 4.8. For each module C and injective coresolution (e,Y) of a module A , there is a natural isomorphism

$$\operatorname{Ext}^n(C,A) \cong H^n(\operatorname{Hom}(C,Y)) \ , \quad n = 0,1,\dots \ .$$

We can summarize these last two theorems in the scheme

$$H^n(\mathrm{Hom}\ (\mathrm{Proj}\ C,A)) \cong \mathrm{Ext}^n\ (C,A) \cong H^n(\mathrm{Hom}\ (C,\mathrm{Inj}\ A))$$

where Proj C denotes an arbitrary projective resolution of C and Inj A an arbitrary injective coresolution of A .

5. Exact Sequence for Ext

We have observed that the functor Hom does not preserve exact sequences. In the case of the second variable the problem was: If A is a submodule of B and G is any module, when can a homomorphism $\gamma : G \to B/A$ be lifted to B ? This problem is answered in

Lemma 5.1. Given a short exact sequence $E = (\kappa,\sigma) : A \rightarrowtail B \twoheadrightarrow C$ of R-modules and module homomorphism, then for any module G a homomorphism $\gamma : G \to C$ can be lifted to a homomorphism $\hat{\gamma} : G \to B$ if and only if the extension $E\gamma$ is zero.

The proof is straightforward.

A short exact sequence $E = (\kappa,\sigma) : A \rightarrowtail B \twoheadrightarrow C$ yields homomorphisms $\kappa_* : \mathrm{Ext}^n\ (G,A) \to \mathrm{Ext}^n\ (G,B)$ and $\sigma_* : \mathrm{Ext}^n\ (G,B) \to \mathrm{Ext}^n\ (G,C)$ given by $\kappa_* S = \kappa S$ and $\sigma_* S = \sigma S$. There also exists a *connecting* homomorphism.

$$E_* : \mathrm{Ext}^n\ (G,C) \to \mathrm{Ext}^{n+1}\ (G,A)$$

defined by left multiplication by E . These homomorphisms give

Theorem 5.2. If $E = (\kappa,\sigma) : A \rightarrowtail B \twoheadrightarrow C$ is a short exact sequence of R-modules, then the sequence

$$0 \to \mathrm{Hom}(G,A) \overset{\kappa_*}{\to} \mathrm{Hom}(G,B) \overset{\sigma_*}{\to} \mathrm{Hom}(G,C) \overset{E_*}{\to} \mathrm{Ext}^1(G,A) \overset{\kappa_*}{\to} \mathrm{Ext}^1(G,B) \to$$

$$\ldots \to \mathrm{Ext}^n(G,B) \overset{\sigma_*}{\to} \mathrm{Ext}^n(G,C) \overset{E_*}{\to} \mathrm{Ext}^{n+1}(G,A) \to \ldots \tag{5.1}$$

of abelian groups is exact for any R-module G .

Proof. Exactness at Hom (G,A) and Hom (G,B) is known. Lemma 5.1 gives exactness at Hom (G,C) .

Exactness at Ext^1 (G,A) . $\kappa_* E_* = (\kappa E)_* = 0$. Suppose $F \in \mathrm{Ext}^1$ (G,A) such that $\kappa F \equiv 0$ in Ext^1 (G,B) . We have a commutative diagram

$$\begin{tikzcd}
F : & A \arrow[r, tail, "\kappa_1"] \arrow[d, "\kappa"'] & B_1 \arrow[r, two heads, "\sigma_1"] \arrow[dl, dashed, "\pi_1\beta"'] \arrow[d, dashed, "\beta"] & G \arrow[d, dash] \arrow[ddll, dashed, "\alpha"] \\
\kappa F : & B \arrow[r, tail, "\iota_1"] \arrow[d, two heads, "\sigma"'] & B+G \arrow[r, two heads, "\pi_2"] & G \\
& C & &
\end{tikzcd}$$

with exact rows and an exact column. $\sigma(\pi_1\beta)\ \kappa_1 = 0$, so there is an induced map $\alpha : B_1/A \cong G \to C$ with $\alpha\sigma_1 = \sigma(\pi_1\beta)$. Then $E\alpha \equiv F$.

In the general case $(n > 0)$, the proof proceeds as follows. Take any free resolution X of G and apply the exact cohomology sequence II (3.2) over the sequence E. Since the cohomology groups $H^n(X;A)$ are $\mathrm{Ext}^n(G,A)$, and so on, this yields an exact sequence with the same terms as (5.1). The proof is completed by showing commutativity in the diagram

$$\begin{array}{ccccccc}
\mathrm{Ext}^n(G,A) & \xrightarrow{\kappa_*} & \mathrm{Ext}^n(G,B) & \xrightarrow{\sigma_*} & \mathrm{Ext}^n(G,C) & \xrightarrow{E_*} & \mathrm{Ext}^{n+1}(G,A) \\
\zeta\downarrow & & \zeta\downarrow & & \downarrow\zeta & & \zeta\downarrow \\
H^n(X;A) & \xrightarrow{\kappa_*} & H^n(X;B) & \xrightarrow{\sigma_*} & H^n(X;C) & \xrightarrow{E_*} & H^{n+1}(X;A)
\end{array}$$

where in the second row the maps are defined by $\kappa_*(\mathrm{cls}\ f) = \mathrm{cls}\ \kappa f$, $\sigma_*(\mathrm{cls}\ g) = \mathrm{cls}\ \sigma g$ and $E_*(\mathrm{cls}\ h) = \mathrm{cls}\ f$ where $\kappa f = (-1)^{n+1}k\partial$ and $\sigma k = h$, and ζ is defined as in Thm. 4.5.

Dually we have

Theorem 5.3. For $E : A \rightarrowtail B \twoheadrightarrow C$ a short exact sequence of R-modules, the sequence

$$0 \to \mathrm{Hom}(C,G) \to \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G) \to \mathrm{Ext}^1(C,G) \to \mathrm{Ext}^1(B,G) \to$$
$$\cdots \to \mathrm{Ext}^n(B,G) \to \mathrm{Ext}^n(A,G) \to \mathrm{Ext}^{n+1}(C,G) \to \cdots$$

of abelian groups is exact for any R-module G.

For abelian groups the sequences of the two previous theorems are reduced according to

Corollary 5.4. For abelian groups we have two exact sequences

$$0 \to \mathrm{Hom}(G,A) \to \mathrm{Hom}(G,B) \to \mathrm{Hom}(G,C) \to \mathrm{Ext}(G,A) \to \mathrm{Ext}(G,B) \to \mathrm{Ext}(G,C) \to 0$$

$$0 \to \mathrm{Hom}(C,G) \to \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G) \to \mathrm{Ext}(C,G) \to \mathrm{Ext}(B,G) \to \mathrm{Ext}(A,G) \to 0 \quad .$$

Proof. We prove exactness in second sequence. Assume $\kappa : A \to B$ is a monomorphism. Obtain the exact sequence $K \rightarrowtail F \xrightarrow{\phi} B$ with F free. Let $L = \phi^{-1}(\kappa A)$. Then ϕ maps L onto A and has kernel K , giving a commutative diagram

$$\begin{array}{lccccc} E_1 : & K & \rightarrowtail & L & \twoheadrightarrow & A \\ & \Big| & & \downarrow & & \downarrow \kappa \\ E_2 : & K & \rightarrowtail & F & \xrightarrow{\phi} & B \end{array} \quad .$$

Thus $E_1 \equiv E_2\kappa$. This yields a commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}\ (K,G) & \xrightarrow{E_2^*} & \mathrm{Ext}\ (B,G) & & \\ \Big| & & \kappa^* \downarrow & & \\ \mathrm{Hom}\ (K,G) & \xrightarrow{E_1^*} & \mathrm{Ext}\ (A,G) & \to & \mathrm{Ext}\ (L,G) \end{array}$$

with bottom row exact. L free implies that $\mathrm{Ext}\ (L,G) = 0$. Hence E_1^* , and therefore κ^* are epimorphisms.

6. Axiomatic Description of Ext.

The properties obtained for the sequence of functors Ext^n suffice to determine those functors up to a natural equivalence, in the following sense.

Theorem 6.1. For each $n = 0, 1, \ldots,$ let there be given a contravariant functor $\mathrm{Ex}^n(A)$ of the module A, taking abelian groups as values, and for each n and each short exact sequence $E : A \rightarrowtail B \twoheadrightarrow C$ let there be given a homomorphism $E^n : \mathrm{Ex}^n(A) \to \mathrm{Ex}^{n+1}(C)$ which is natural for morphisms $\Gamma : E \to E'$ of short exact sequences. Suppose that there is a fixed module G such that

$$\mathrm{Ex}^0(A) = \mathrm{Hom}(A,G) \qquad \text{for all } A$$

$$\mathrm{Ex}^n(F) = 0 \quad n > 0 \quad \text{and all free } F,$$

and suppose that for each $E = (\kappa,\sigma)$ the sequence

$$\ldots \quad \mathrm{Ex}^n(C) \xrightarrow{\sigma^*} \mathrm{Ex}^n(B) \xrightarrow{\kappa^*} \mathrm{Ex}^n(A) \xrightarrow{E^*} \mathrm{Ex}^{n+1}(C) \to \ldots$$

is exact. Then there is for each A and n an isomorphism $\phi_A^n : \mathrm{Ext}^n(A) \cong \mathrm{Ext}^n(A,G)$ with $\phi_A^0 = 1$, which is natural in A and such that the diagram

$$\begin{array}{ccc} Ex^n\ (A) & \overset{E^n}{\rightarrow} & Ex^{n+1}\ (C) \\ \downarrow \phi^n & & \downarrow \phi^{n+1} \\ Ext^n\ (A,G) & \overset{E^*}{\rightarrow} & Ext^{n+1} (C,G) \end{array}$$

is commutative for all n and all short exact $E : A \rightarrowtail B \twoheadrightarrow C$.

Proof. MacLane [8] pp. 99 - 101 .

The dual characterization for $Ext\ (C,A)$ as a functor of A using the exact sequence (5.1) is now stated.

Theorem 6.2. For a fixed G , the covariant functors $Ext^n\ (G,A)$ of A , $n = 0$, 1, ... together with the natural homomorphisms $E_* : Ext^n\ (G,C) \rightarrow Ext^{n+1}\ (G,A)$ defined for short exact sequences E of modules, are characterized up to a natural isomorphism by these three properties:

$Ext^o\ (G,A) = Hom\ (G,A)$ for all A ,

$Ext^n\ (G,Q) = 0$ for $n > 0$ and all injective Q .

The sequence (5.1) is exact for all E .

CHAPTER IV

TORSION PRODUCTS AND HOMOLOGY

The dual of the extension functor, the torsion functor, is now briefly discussed in relation to its significance in homology. First, the torsion product, $\mathrm{Tor}_n\,(G,A)$, of two modules is defined and shown to be $H_n(X;A)$ (the n^{th} homology group of the complex $X \otimes A$) where X is an arbitrary projective resolution of G . $\mathrm{Tor}_n\,(G,A)$ is also shown to be $H_n\,(X \otimes Y)$ where X and Y are projective resolutions of G and A , respectively. The Künneth formulas are developed and Tor_1 is used to calculate the homology of a complex of torsion free abelian groups over a group from the homology of the complex; this being the universal coefficient theorem for homology.

1. Duality

The dual of a left R-module A was defined in I §6 to be the right R-module $A^* = \mathrm{Hom}\,(A,R)$. We need two results from duality theory for the development of torsion products. The proof of these two theorems may be found in MacLane [8] pp. 146-147 .

Theorem 1.1. If L is a finitely generated and projective left R-module, then L^* is a finitely generated and projective right R-module. For such L , there is a natural isomorphism

$$\phi : L \to L^{**} = \mathrm{Hom}\ (\mathrm{Hom}\ (L,R)\ ,\ R)$$

with $(\phi a)\, f = f(a)$.

Theorem 1.2. If L is a finitely generated projective left module and C an arbitrary left module, there is a natural isomorphism

$$\eta : L^{*} \otimes C \to \mathrm{Hom}\ (L,C) \tag{1.1}$$

with $\eta(f \otimes c)\, a = f(a)\, c$.

Finally if F is free on generators $e_1, \ldots, e_n$, then F^{*} is free on generators $e^1, \ldots, e^n$ defined by

$$e^j(e_i) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} .$$

2. <u>Torsion Products</u>

For fixed $n \geq 0$ we consider chain complexes L of length n

$$L : L_n \to L_{n-1} \to \ldots \to L_1 \to L_o$$

where each L_k is a finitely generated projective right R-module. The dual L^{*}

$$L^* : L_o^* \to L_1^* \to \ldots \to L_{n-1}^* \to L_n^*$$

is also a chain complex. In L^* each L_k^* is a finitely generated projective left R-module.

Given R-modules G_R and ${}_RC$ we define a set Tor_n (G,C) whose objects are all the triples

$$t = (\mu, L, \nu)$$

where L is a chain complex of length n of finitely generated projective right R-modules while $\mu : L \to G$ and $\nu : L^* \to C$ are chain transformations. If L' is a second such complex and $\rho : L \to L'$ a chain transformation, then so is the dual $\rho^* : L'^* \to L^*$. Given $\mu' : L' \to G$ and $\nu' : L^* \to C$ we propose that

$$(\mu'\rho, L, \nu) = (\mu', L', \nu\rho^*) \tag{2.1}$$

The set Tor_n becomes a covariant bifunctor when for $\eta : G \to G'$ and $\zeta : C \to C'$ we set

$$\eta_*(\mu, L, \nu) = (\eta\mu, L, \nu)$$
$$\text{and} \quad \zeta_*(\mu, L, \nu) = (\mu, L, \zeta\nu) \quad .$$

The direct sum of two triples t_1 and t_2 in $\mathrm{Tor}_n(G,C)$ is given by

$$(\mu_1,L^1,\nu^1) \oplus (\mu_2,L^2,\nu^2) = (\mu_1 \oplus \mu_2, L^1 \oplus L^2, \nu_1 \oplus \nu_2)$$

an element in $\mathrm{Tor}_n\,(G \oplus G,\ C \oplus C)$. If ω_G is the automorphism of $G \oplus G$ given by $\omega(g_1,g_2) = (g_2,g_1)$. Then $(\omega_G)_*(t_1 \oplus t_2) = (\omega_C)_*(t_2 \oplus t_1)$ as one sees by applying (2.1) with $\rho : L^1 \oplus L^2 \to L^2 \oplus L^1$ the map interchanging the summands.

Theorem 2.1. For R-modules G_R and ${}_RC$ the set $\mathrm{Tor}_n\,(G,C)$ is an abelian group under additon given by

$$t_1 + t_2 = (\nabla_G)_*(\nabla_C)_*(t_1 \oplus t_2) \ .$$

Proof. The associativity law follows from the associativity of the codiagonal maps. The commutativity follows from $(\omega_G)_*(t_1 \oplus t_2) = (\omega_C)_*(t_2 \oplus t_1)$ and $\nabla_G\omega_G = \nabla_G$. The identity is $(0,0,0) \in \mathrm{Tor}_n\,(G,C)$, where the middle zero designates the zero complex of length n . The inverse of $t = (\mu,L,\nu)$ is $-t = (-\mu,L,\nu)$.

Proposition 2.2. The symbols (μ,L,ν) in Tor_n are additive in μ and ν ; e.g.,

$$(\mu_1 + \mu_2,L,\nu) = (\mu_1,L,\nu) + (\mu_2,L,\nu)$$

This follows from the fact that the dual of the diagonal map $(\Delta_L)_* : L \to L \oplus L$ is the codiagonal map $\nabla_{L^*} : L^* \oplus L^* \to L$, and using $\mu_1 + \mu_2 = \nabla_G(\mu_1 \oplus \mu_2)\,\Delta_C$.

Proposition 2.3. Every element of $\mathrm{Tor}_n(G,C)$ has the form (μ,F,ν) where F is a chain complex of finitely generated free right modules.

Proof. Since every free complex F is projective, each element (μ,F,ν) is in $\mathrm{Tor}_n(G,C)$. Take any $(\mu,L,\nu) \in \mathrm{Tor}_n$. Each L_k can be written as a direct summand of some finitely generated free module $F_k = L_k \oplus M_k$. Make F_k a complex with boundary $\partial \oplus 0$. The injection $\iota : L \to F$ and the projection $\pi : F \to L$ are chain transformations with $\pi\iota = 1$ and $\iota^*\pi^* = 1$. By our equality rule (2.1)

$$(\mu,L,\nu) = (\mu,L,\nu\iota^*\pi^*) = (\mu\pi,F,\nu\iota^*) .$$

For $n = 0$, Tor_o may be identified with $\otimes$.

Theorem 2.4. There is a natural isomorphism

$$\psi : G \otimes C \cong \mathrm{Tor}_o(G,C)$$

Proof: Each $g \in G$ determines a map $\mu_g : R \to G$ by $\mu_g r = gr$ while each $c \in C$ determines a map $\nu_c : R \cong R^* \to C$ by $\nu_c 1 = c$. This gives a function $G \times C \to \mathrm{Tor}_o(G,C)$ with $(g,c) \to (\mu_g, R, \nu_c)$. This function is bilinear. Hence there is a group homomorphism

$$\psi : G \otimes C \to \mathrm{Tor}_o(G,C)$$

with $\psi(g \otimes c) = (\mu_g, R, \nu_c)$. It is natural . This homomorphism extends linearly and gives for $\sum g_i \otimes c_i \in G \otimes C$ with $g_i = 0$, $i \neq 1,\ldots,n$, $\psi(\sum g_i \otimes c_i) = (\mu, F, \nu)$ where F is free on a generating set of n elements $e_1,\ldots,e_n$, $\mu(e_i) = g_i$ and $\nu(e^i) = c_i$.

For the inverse map θ , we write $t \in \mathrm{Tor}_o\,(G,C)$ as $t = (\mu,F,\nu)$ where F is free on generators $e_1,\ldots,e_n$, say. Set

$$\theta(\mu,F,\nu) = \sum \mu(e_i) \otimes \nu(e^i) .$$

θ is well defined and is independent of a choice of a particular base for F . It is easily seen that θ is a two sided inverse of ψ .

Corollary 2.5. For L a finitely generated projective right R-module there exists a natural isomorphism

$$\xi : \mathrm{Hom}\,(L^*,C) \cong L \otimes C$$

defined by $\xi(\nu) = (1_L, L, \nu)$.

Proof. By additivity (Prop. 2.2) ξ is a homomorphism. The proof of naturality is straightforward. To show it an isomorphism it suffices to prove the composite

$$L \otimes C \overset{\sim}{=} L^{**} \otimes C \overset{\eta}{\cong} \mathrm{Hom}\,(L^*,C) \to \mathrm{Tor}_o\,(L,C) \overset{\sim}{=} L \otimes C$$

an isomorphism where η is defined as in (1.1) . For $L = F$,

finitely generated and free, the definitions of the involved maps clearly give the composition as the identity. Any L is a direct summand of a finitely generated free module F. By naturality of the involved maps, direct summands are mapped to direct summands. Hence the composition is the identity for any L.

From the above we see that any element t of $\mathrm{Tor}_o\,(L,C)$ has a unique representation as $t = (1_L, L, \nu)$ for some $\nu : L^* \to C$.

3. Torsion Products by Resolutions

$\mathrm{Ext}^n\,(C,A)$ may be calculated from a projective resolution X of C as $H^n\,(\mathrm{Hom}\,(X,A))$. There is an analogous calculation for $\mathrm{Tor}_n\,(G,A)$. If (X,e) is a projective resolution over G, then $\{X \otimes A\ ,\ \partial \otimes 1\}$ is a complex of abelian groups. The comparison theorem together with II Cor. 1.2 shows the homology $H_n(X \otimes A)$ independent of the choice of X, hence is a functor of G and A. This homology group is now identified with $\mathrm{Tor}_n\,(G,A)$.

Theorem 3.1. For a resolution (X,e) over the module G_R and a module ${}_RA$, there is a homomorphism

$$\omega : \mathrm{Tor}_n\,(G,A) \to H_n(X \otimes A) \qquad n = 0,1,\ldots \tag{3.1}$$

natural in A.

Proof. We have a natural isomorphism $X_n \otimes A \cong \mathrm{Tor}_o\,(X_n,A)$, so

we define $\omega : \mathrm{Tor}_n\,(G,A) \to H_n(\mathrm{Tor}_o\,(X,A))$. For $t = (\mu,L,\nu)$ $\varepsilon\ \mathrm{Tor}_n\,(G,A)$, the comparison theorem yields a chain transformation $h : L \to X$ lifting 1_G (via e) . Set

$$\omega(t) = \mathrm{cls}\,(h_n,L_n,\nu) \quad .$$

Clearly $(h_n,L_n,\nu)\ \varepsilon\ \mathrm{Tor}_o\,(X_n,A)$. It is easily checked that it is a cycle there and that the homology class of this cycle is unique. Furthermore if $t = (\mu'\rho,L,\nu)$ and $t' = (\mu',L',\nu\rho^*)$ for some $\rho : L \to L'$ are equal according to (2.1) while $h' : L' \to X$, then $h'\rho : L \to X$ and $\omega t = \omega t'$. Hence ω is well defined.

To show ω is a group homomorphism we note that two chain transformations $h^i : L^i \to X$ yield the chain transformation $\nabla_X(h^1 \oplus h^2) : L^1 \oplus L^2 \to X$ and hence

$$\begin{aligned}\omega[(\mu_1,L^1,\nu_1) + (\mu_2,L^2,\nu_2)] &= \omega(\mu_1 \oplus \mu_2,L^1 \oplus L^2,\nu_1 \oplus \nu_2)\\ &= \mathrm{cls}[\nabla(h_n^1 \oplus h_n^2),L_n^1 \oplus L_n^2,\nabla(\nu_1 \oplus \nu_2)]\\ &= \omega(\mu_1,L^1,\nu_1) + \omega(\mu_2,L^2,\nu_2) \ .\end{aligned}$$

Naturality in A is immediate.

Corollary 3.2. If X is a projective resolution, ω is an isomorphism, natural in G and A .

Proof. Naturality in G follows by observing that a chain

transformation $f : X \to X'$ lifting $\eta : G \to G'$ composes with an $h : L \to X$ to give $fh : L \to X'$.

It suffices to show ω an isomorphism when X is free. Any homology class in $X \otimes A = \mathrm{Tor}_o(X,A)$ is the class of a cycle in $X' \otimes A = \mathrm{Tor}_o(X',A)$ where X' is a suitable finitely generated subcomplex of X . Cor. 2.5 gives this cycle as $(1,X'_n,\nu)$ for some $\nu : X'^*_n \to A$. If the complex X' with $e' : A' \to G$ is cut off beyond dimension n , then $t = (e',X',\nu) \in \mathrm{Tor}_n(G,A)$. The injection $\iota : X' \to X$ gives $\omega t = \mathrm{cls}\,(\iota,X'_n,\nu)$. Hence ω is an epimorphism.

Suppose $\omega t = 0$, for some t . Thus (h_n,L_n,ν) is a boundary in $\mathrm{Tor}_o(X,A)$ hence also in $\mathrm{Tor}_o(X',A)$ for $X' \subseteq X$ a finitely generated free subcomplex. Choose X' to contain $h(L)$ and write h' for $h : L \to X'$. Then $(h'_n,L_n,\nu) = (1,X'_n,\nu h'^*_n)$ the boundary of some $(n + 1)$-chain of $\mathrm{Tor}_o(X',A)$. By Cor. 2.5, write this chain as $(1,X'_{n+1},\xi)$ for some $\xi : X'^*_{n+1} \to A$.

Now

$$(1,X'_n,\nu h'^*_n) = \partial(1,X'_{n+1},\xi) = (1,X'_n,\xi\partial^*)$$

and the uniqueness of Cor. 2.5 yields $\nu h'^*_n = \xi\partial^*$. Let ${}^n_oX'$ be the part of X' from X'_o to X'_n inclusive and ${}^{n+1}_{1}X'$ the part from X'_1 to X_{n+1} so that $h' : L \to {}^n_oX'$ and $\partial : {}^{n+1}_{1}X' \to {}^n_oX'$ are chain transformations. Thus

$$\begin{aligned} t = (\mu,L,\nu) = (e'h',L,\nu) &= (e',{}^{n}_{o}X',\nu h'^{*}) \\ = (e',{}^{n}_{o}X',\xi\partial^{*}) &= (e'\partial,{}^{n+1}_{1}X',\xi) \\ = (0,\ -,\ -) &= 0 \quad . \end{aligned}$$

For an exact sequence E of modules and X a projective resolution, $X \otimes E$ is an exact sequences of complexes and chain transformations. The exactness of the homology sequence for $X \otimes E$ together with the above isomorphism yields

Theorem 3.3. A short exact sequence $E : A \rightarrowtail B \twoheadrightarrow C$ of left R-modules and a right R-module G yield a long exact sequence

$$\ldots \to \mathrm{Tor}_n(G,A) \to \mathrm{Tor}_n(G,B) \to \mathrm{Tor}_n(G,C) \to \mathrm{Tor}_{n-1}(G,A) \to \ldots \qquad (3.2)$$

of abelian groups ending with $\mathrm{Tor}_o\ (G,C) \cong G \otimes C \to 0$.

For a projective module $A = P$, the exactness of a resolution X makes $H_n(X \otimes P)$ and hence $\mathrm{Tor}_n(G,P)$ zero for $n > 0$. We can now characterize Tor by axioms as we did for Ext in III § 6 , as follows.

Theorem 3.4. For a fixed right module G the covariant functors $\mathrm{Tor}_n\ (G,A)$ of A , $n = 0, 1, \ldots,$ taken together with the connecting homomorphisms $\mathrm{Tor}_n\ (G,C) \to \mathrm{Tor}_{n-1}\ (G,A)$, natural for short exact sequences E of modules, are characterized up to a natural isomorphism by the properties

i. $\mathrm{Tor}_o\,(G,A) = G \otimes A$ for all A ,

ii. $\mathrm{Tor}_n\,(G,F) = 0$ for $n > 0$ and all free F ,

iii. The sequence (3.2) is exact for all E .

Theorem 3.5. The following properties of a right R-module G are equivalent:

i. $\mathrm{Tor}_1\,(G,C) = 0$ for all C

ii. If $\kappa : A \to B$ is a monomorphism then so is $1 \otimes \kappa$: $G \otimes A \to G \otimes B$.

iii. Every exact sequence of modules remains exact upon tensor multiplication by G .

iv. If $G' \rightarrowtail G'' \twoheadrightarrow G$ is exact and A any module, then $G' \otimes A \rightarrowtail G'' \otimes A \twoheadrightarrow G \otimes A$ is exact.

v. $\mathrm{Tor}_n\,(G,C) = 0$ for $n > 0$ and all C .

The proof is straightforward. A module with these properties is said to be *flat*. Clearly every projective module is flat.

4. Torsion Product of Groups

For abelian groups (Z-modules) G and A $\mathrm{Tor}_1(G,A)$, written $\mathrm{Tor}\,(G,A)$, is the set of triples (μ,L,ν) where $\mu : L \to G$, $\nu : L^* \to A$, $L : L_1 \to L_o$, L_1 and L_o are finitely generated projective (i.e. free) abelian groups. Thus L is the direct sum of a finite number of isomorphic copies of Z . Hence we may consider $L : Z \to Z$. The triple (μ,L,ν) determines three elements

$$Z \xrightarrow{\partial} Z \xrightarrow{\mu} G \qquad\qquad Z^* \overset{\sim}{=} Z \xrightarrow{\nu} A$$

by $\mu(1) = g$, $\nu(1) = a$ and $\partial(1) = m$. Since $\mu\partial = 0$ and $\nu\partial = 0$ we have $mg = 0 = ma$. The triple (μ,L,ν) has become a triple $[g,m,a]$ where $mg = 0 = ma$.

It is easily seen that the equality in Tor $(\mu'\rho,L,\nu) = (\mu,L',\nu\rho^*)$ for $\rho : L \to L'$ gives

$$[n_o g',m,a] = [g',m',n_1 a]$$

where $n_o m = m'n_1$ and $g'm' = 0 = ma$.

The additivity of (μ,L,ν) in μ and ν gives the additivity of $[g,m,a]$ in g and a .

Thus the torsion product of two abelian groups G and A , Tor (G,A) , may be defined as the abelian group generated by the set of all elements $[g,m,a]$ with $m \in Z$, $mg = 0 = ma$, subject to the relations

$$
\begin{array}{lll}
[g_1 + g_2,m,a] = [g_1,m,a] + [g_2,m,a] & & mg_i = 0 = ma \\
[g,m,a_1 + a_2] = [g,m,a_1] + [g,m,a_2] & & mg = 0 = ma_i \\
[g,mn,a] = [mg,n,a] & & mng = 0 = na \\
[g,mn,a] = [g,m,na] & & mg = 0 = mna \ .
\end{array}
$$

The first two relations imply that $[0,m,a] = 0 = [g,m,0]$

and hence that $\mathrm{Tor}\,(G,A) = 0$ when G is torsion free. Since $\mathrm{Tor}\,(G_1 \oplus G_2, A) \cong \mathrm{Tor}\,(G_1,A) \oplus \mathrm{Tor}\,(G_2,A)$, to calculate $\mathrm{Tor}\,(G,A)$ for G finitely generated, it is sufficient to make a calculation for G finite cyclic. For $G = Z_q(g)$ a cyclic group of order q and generator g there is an isomorphism

$$\mathrm{Tor}\,(Z_q(g)\;,\;A) \cong {}_qA = \{a;\; qa = 0\}$$

given by $[kg,m,a] \to (mk/q)a$, $(a \to [g,q,a])$ natural in A.

For abelian groups the exact sequence of (3.2) is reduced as follows.

Theorem 4.1. For an abelian group G and an exact sequence of abelian groups $E = (\kappa,\sigma) : A \rightarrowtail B \twoheadrightarrow C$,

$$\begin{aligned} 0 \to \mathrm{Tor}(G,A) \xrightarrow{\kappa_*} \mathrm{Tor}(G,B) \to \mathrm{Tor}(G,C) \to G \otimes A \\ \to G \otimes B \to G \otimes C \to 0 \end{aligned}$$

is an exact sequence of abelian groups.

Proof: By Thm. 3.3 it remains to prove κ_* a monomorphism. First consider G finitely generated. Assuming G has a minimal generating set of n elements $g_1, g_2, \ldots, g_n$ we have

$$G \cong \sum_{i=1}^{s} [g_i] \oplus \sum_{i=s+1}^{n} [g_i]$$

where the first summand is a torsion group and the second torsion free. Thus $\mathrm{Tor}\,(G,A) \cong \sum \mathrm{Tor}\,([g_i],A)$, $1 \leq i \leq s$. The problem reduces to showing $\mathrm{Tor}\,([g_i],A) \to \mathrm{Tor}\,([g_i],B)$ is a monomorphism where $[g_i]$ is cyclic of order, say q . $\mathrm{Tor}\,([g_i],A) \cong {}_qA$ and ${}_qA \to {}_qB$ is clearly a monomorphism. Thus for G finitely generated κ_* is a monomorphism.

In general an element $u = \sum [g_i,m_i,a_i] \in \mathrm{Tor}\,(G,A)$ involves only a finite number of elements of G say $g_1, \ldots, g_n$. If its image $\kappa_* u = \sum [g_i,m_i,a_i]$ is zero in $\mathrm{Tor}\,(G,B)$, it is zero because of a finite number of elements of G say $h_1, \ldots, h_m$. Let G_o be the subgroup of G generated by $g_1, \ldots, g_n$, $h_1, \ldots, h_m$, and $\iota : G_o \to G$ the inclusion. Then $u \in \mathrm{Tor}\,(G_o,A)$ and $\iota_* u = u$. By naturality the diagram

$$\begin{array}{ccccc} 0 & \to & \mathrm{Tor}\,(G_o,A) & \xrightarrow{H_*} & \mathrm{Tor}\,(G_o,B) \\ & & \iota_* \downarrow & & \downarrow \\ 0 & \to & \mathrm{Tor}\,(G,A) & \to & \mathrm{Tor}\,(G,B) \end{array}$$

commutes. The first row is exact as G_o is finitely generated. Thus $\kappa_* u = 0$ gives $u = 0$.

5. Tensor Product of Complexes

If K_R and ${}_RL$ are chain complexes of right and left R-modules, respectively, then their *tensor product* $K \otimes L$ is the chain complex of abelian groups with

$$(K \otimes L)_n = \sum_{p+q=n} K_p \otimes L_q \tag{5.1}$$

with boundary homomorphism defined on generators $k \otimes \ell \in K_p \otimes L_q$ by

$$\partial(k \otimes \ell) = \partial k \otimes \ell + (-1)^p k \otimes \partial\ell \quad . \tag{5.2}$$

If K and L are positive complexes, so is $K \otimes L$, and the direct sum above is finite, with p running from 0 to n.

For chain transformations $f : K \to K'$ and $g : L \to L'$, the definition $(f \otimes g)(k \otimes \ell) = fk \otimes g\ell$ gives a chain transformation $K \otimes L \to K' \otimes L'$. In this way the tensor product of complexes is a covariant bifunctor.

As a first application of the tensor product of complexes, we show that the torsion products can be computed from resolutions of both arguments as follows.

Theorem 5.1. If X is a projective resolution over (G,e) and Y a projective resolution over (A,η) of the R-modules G_R and ${}_RA$, then $e \otimes 1 : X \otimes Y \to G \otimes Y$ induces an isomorphism $H_n(X \otimes Y) \cong H_n(G \otimes Y)$ and hence an isomorphism

$$H_n(X \otimes Y) \cong \mathrm{Tor}_n(G,A) \qquad n = 0, 1, \ldots \quad .$$

Proof. Let F^k, $k = 0, 1, \ldots$, be the subcomplex of $X \otimes Y$

spanned by all $X_i \otimes Y_j$ with $j \leq k$, while M^k is the subcomplex of $G \otimes Y$ consisting of all $G \otimes Y_j$ with $j \leq k$. Then

$$0 = F^{-1} \subseteq F^o \subseteq F^1 \subseteq \ldots \subseteq X \otimes Y$$

$$0 = M^{-1} \subseteq M^o \subseteq M^1 \subseteq \ldots \subseteq G \otimes Y \ .$$

By the comparison theorem lift $1_{G \otimes A}$ to a chain transformation $f : X \otimes Y \to G \otimes Y$ with $f_o = e \otimes 1$. f depends only on $e \otimes 1$, so we say that $e \otimes 1$ is a chain transformation $X \otimes Y \to G \otimes Y$. Clearly $e \otimes 1$ maps F^k into M^k.

Consider the quotient complexes F^k/F^{k-1} and M^k/M^{k-1}

$$(F^k/F^{k-1})_n = \begin{cases} 0 & n < k \\ X_o \otimes Y_k & n = k \\ X_n \otimes Y_k & n > k \end{cases}, \qquad (M^k/M^{k-1})_n = \begin{cases} 0 & n < k \\ G \otimes Y_k & n = k \\ 0 & n > k \end{cases}.$$

Because each Y_k is projective and $\ldots \to X_1 \to X_o \to G \to 0$ is exact the complex

$$F^k/F^{k-1} : \ldots \to X_n \otimes Y_k \to \ldots \to X_o \otimes Y_k \to G \otimes Y_k \to 0$$

is exact. Then $e \otimes 1 : F^k/F^{k-1} \to M^k/M^{k-1}$ induces an isomorphism in homology for all k. Also $e \otimes 1$ maps the exact sequence $F^{k-1} \rightarrowtail F^k \twoheadrightarrow F^k/F^{k-1}$ into the corresponding exact sequence for the M's. This gives the following commutative diagram

in homology in which both rows are exact

$$\begin{array}{ccccccccc} H_{n+1}(F^k/F^{k-1}) & \to & H_n(F^{k-1}) & \to & H_n(F^k) & \to & H_n(F^k/F^{k-1}) & \to & H_{n-1}(F^{k-1}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{n+1}(M^k/M^{k-1}) & \to & H_n(M^{k-1}) & \to & H_n(M^k) & \to & H_n(M^k/M^{k-1}) & \to & H_{n-1}(M^{k-1}) \end{array} .$$

We claim $H_n(F^k) \to H_n(M^k)$ is an isomorphism for all n , k . This is true for $k < 0$ and for all n . Assume true for all smaller k and all n . Then the four outside vertical maps are isomorphisms and hence the five lemma gives $H_n(F^k) \cong H_n(M^k)$.

In dimension n every cycle or boundary in $X \otimes Y$ will appear in F^{n+1} as $(X \otimes Y)_n = F_n^{n+1}$. Thus the isomorphism $H_n(F^k) \cong H_n(M^k)$ for $k \geq n+1$ gives $H_n(X \otimes Y) \cong H_n(F^k)$. Now $H_n(M^k) \cong H_n(G \otimes Y) \cong \mathrm{Tor}_n(G,A)$. Hence $H_n(X \otimes Y) \cong \mathrm{Tor}_n(G,A)$.

A sequence of subcomplexes F^k of $X \otimes Y$ as in the above theorem is called a *filtration* of $X \otimes Y$.

6. The Künneth Formulas

The tensor product of complexes corresponds to the cartesian product of spaces X and Y in the sense that the singular complex $C(X \times Y)$ can be proven (MacLane, [8], VIII 8) chain equivalent to $C(X) \times C(Y)$. The problem of this section is to determine the homology of $K \otimes L$ in terms of K and L .

The boundary formula (5.2) shows that the tensor product

$u \otimes v$ of two cycles is a cycle in $K \otimes L$ while the tensor product of a cycle and a boundary is a boundary. Thus for cycles u and v in K and L, respectively,

$$p(\mathrm{cls}\, u \otimes \mathrm{cls}\, v) = \mathrm{cls}(u \otimes v)$$

is a well defined homology class in $K \otimes L$, so yields a homomorphism

$$p : H_m(K) \otimes H_q(L) \to H_{m+q}(K \otimes L) .$$

This gives a homomorphism

$$p : \sum_{m+q=n} H_m(K) \otimes H_q(L) \to H_n(K \otimes L) ,$$

which under certain conditions on $B_m(K)$, $C_m(K)$ and $H_m(K)$, i.e. the boundaries, cycles and homology classes, respectively, of K, give all of $H_n(K \otimes L)$. The homomorphism p is termed the *homology product*.

Theorem 5.1. (The Künneth Formula) If L is a complex of left R-modules, while K is a complex of right R-modules with $C_n(K)$ and $B_n(K)$ flat for all n, then there is in each dimension n a short exact sequence

$$\sum_{m+q=n} H_m(K) \otimes H_q(L) \overset{p}{\rightarrowtail} H_n(K \otimes L) \overset{\beta}{\twoheadrightarrow} \sum_{m+q=n-1} \mathrm{Tor}_1(H_m(K), H_q(L)) \quad (5.1)$$

where p is the homology product and β a natural homomorphism.

Before proving Theorem 5.1 we prove a short lemma.

Lemma 5.2. If G is a flat right R-module, $p : G \otimes H_n(L) \cong H_n(G \otimes L)$.

Proof. To say that $H_n(L) = C_n(L)/B_n(L)$ is the n^{th} homology group of L is to say that the diagram

$$\begin{tikzcd} & 0 & 0 & & \\ 0 \arrow[r] & B_n(L) \arrow[u] \arrow[r] & C_n(L) \arrow[u] \arrow[r] & H_n(L) \arrow[r] & 0 \\ & L_{n+1} \arrow[u] \arrow[r, "\partial"] & L_n \arrow[u] \arrow[d, "\partial"] & & \\ & & L_{n-1} & & \end{tikzcd}$$

is commutative with exact rows and columns. Since G is flat, tensoring the diagram with it will produce another exact diagram, which gives the desired isomorphism.

Proof (of Theorem 5.1). Regard $C_n = C_n(K)$, $D_n = K_n/C_n \cong B_{n-1}(K)$ as complexes of flat modules with zero boundary, so that $C \rightarrowtail K \twoheadrightarrow D$ is an exact sequence of complexes. D_n flat by Thm. 3.5 (iv) gives $E : C \otimes L \rightarrowtail K \otimes L \twoheadrightarrow D \otimes L$ as an exact sequence of complexes. The usual exact homology sequence for E reads

$$H_{n+1}(D \otimes L) \xrightarrow{E_{n+1}} H_n(C \otimes L) \to H_n(K \otimes L) \to H_n(D \otimes L) \xrightarrow{E_n} H_{n-1}(C \otimes L)$$

with connecting homomorphisms E_n . Equivalently the sequence

$$0 \to \operatorname{Coker} E_{n+1} \to H_n(K \otimes L) \to \operatorname{Ker} E_n \to 0 \tag{5.2}$$

is exact. We compose this sequence with (5.1) .

The homology modules $H_m(K)$ give the short exact sequence $S : D_{m+1} \rightarrowtail C_m \twoheadrightarrow H_m(K)$. Tensor this with $H_m(L)$. C_m flat gives $\operatorname{Tor}_1 (C_m, H_q(L)) = 0$ and hence the long exact homology sequence summed over $m + q = n$ becomes

$$\begin{array}{ccccccc}
\sum \operatorname{Tor}_1(H_m(K),H_g(L)) & \rightarrowtail & \sum D_{m+1} \otimes H_q(L) & \xrightarrow{\partial' \otimes 1} & \sum C_m \otimes H_q(L) & \twoheadrightarrow & \sum H_m(K) \otimes H_q(L) \\
 & & \downarrow p & & \downarrow p & & \\
 & & H_{n+1}(D \otimes L) & \xrightarrow{E_{n+1}} & H_n(C \otimes L) & &
\end{array}$$

where ∂' denotes the map $D_{m+1} \to C$ induced by ∂ and E_{n+1} the homology connecting homomorphism. The square is commutative while the previous lemma gives both vertical maps isomorphisms. Hence

$$\operatorname{Ker} E_{n+1} \cong \operatorname{Ker} \partial' \otimes 1 \cong \sum \operatorname{Tor}_1 (H_m(K), H_q(L))$$

and $$\operatorname{Coker} E_{n+1} \cong \operatorname{Coker} \partial' \otimes 1 \cong \sum H_m(K) \otimes H_q(L) \ .$$

Thus the sequence (5.2) becomes the desired sequence (5.1) .

$H_m = C_m/B_m$. H_m projective implies that $C_m \twoheadrightarrow A_m$ splits and hence B_n is a direct summand of the projective module H_m and hence is itself projective and consequently flat. H_m flat implies

that $\mathrm{Tor}_1(H_m, H_q) = 0$. This gives an important corollary.

Corollary 5.3. (The Künneth Tensor Formula.) If L is a complex of left R-modules while K is a complex of right R-modules with $C_n(K)$ and $H_n(K)$ projective for all n , then the homology product is an isomorphism

$$p : \sum H_m(K) \otimes H_q(L) \cong H_n(K \otimes L) \quad \text{for all } n .$$

For complexes of abelian groups we can say more.

Theorem 5.4. (The Künneth Formula for Abelian Groups) For chain complexes K and L of abelian groups with K_n torsion free, (5.1) is exact and splits by a non natural homomorphism.

Proof. K_n torsion free implies C_n and B_n are torsion free and hence flat. Thus Thm. 5.1 gives the exactness of the sequence. To show sequence splits we first consider the case when K and L are complexes of free abelian groups. In this case all their subgroups are free. Thus $D_m \cong K_m/C_m$ gives $K_m \cong D_m \oplus C_m$, and hence the homomorphism $\mathrm{cls} : C_m \to H_m(K)$ can be lifted to a homomorphism $\phi_m : K_m \to H_m(K)$ with $\phi_m(c) = \mathrm{cls}\ c$ for each cycle c . There is a similar homomorphism $\psi_q : L_q \to H_q(L)$ for the complex L . The tensor product of these groups homomorphisms yields a map

$$\phi \otimes \psi : K_m \otimes L_q \to H_m(K) \otimes H_q(L) .$$

Since ϕ and ψ vanish on boundaries so does $\phi \otimes \psi$. There is thus an induced map

$$(\phi \otimes \psi)_* : H_n(K \otimes L) \to \sum H_m(K) \otimes H_q(L) .$$

For cycles u and v we have $(\phi \otimes \psi)_* \, p\,(\text{cls } u \otimes \text{cls } v)$ $= (\phi \otimes \psi)_* \text{ cls } (u \times v) = \phi u \otimes \psi v = \text{cls } u \otimes \text{cls } v$. Thus $(\phi \otimes \psi)_*$ is a left inverse of p, splitting our sequence.

Consider complexes K and L with K torsion free. Below we show that we can choose free complexes K' and L' and chain transformations $f : K' \to K$ and $g : L' \to L$ such that $f_* : H_p(K') \cong H_p(K)$ and $g_* : H_q(L') \cong H_q(L)$. The naturality of p and β makes the diagram

$$\begin{array}{ccccccccc}
0 \to & \sum H(K') \otimes H(L') & \to & H(K' \otimes L') & \to & \sum \mathrm{Tor}(H(K'),H(L')) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & \sum H(K) \otimes H(L) & \to & H(K \otimes L) & \to & \sum \mathrm{Tor}(H(K),H(L)) & \to 0
\end{array}$$

commutative. Since the outside vertical maps are isomorphisms, the short five lemma gives the middle one as an isomorphism also. The split exactness of the top sequence implies the split exactness of the bottom one.

Lemma 5.5. If K is a complex of abelian groups there exists a complex X of free abelian groups and a chain transformation $f : X \to K$ such that $f_* : H_n(X) \to H_n(K)$ is an isomorphism for all n.

Proof. It suffices to take X as the direct sum of complexes $X^{(n)}$ with chain transformations $f^{(n)} : X^{(n)} \to K$ such that $(f^{(n)})_* : H_n(X^{(n)}) \cong H_n(K)$, $H_q(X^{(n)}) = 0$, $q \neq n$. For fixed n construct a diagram

$$\begin{array}{ccccc} 0 \to R_{n+1} & \xrightarrow{j} & F_n \to 0 \\ \downarrow \eta & & \downarrow \zeta \\ K_{n+1} & \xrightarrow{\partial} & K_n \end{array} \quad .$$

First write the group C_n of n-cycles of K as a quotient of a free group F_n ; giving $\zeta : F_n \to C_n \subseteq K_n$. Take $R_{n+1} = \zeta^{-1} B_n$ and $j : R_{n+1} \to F_n$ the injection. Since R_{n+1} is free and $\zeta j\, R_{n+1} = K_{n+1}$, ζj lifts to a map η making the diagram commutative. The top row is a complex $X^{(n)}$ with homology $F_n/R_{n+1} \cong C_n/B_n \cong H_n(K)$ in dimension n and zero elsewhere. The vertical maps constitute a chain transformation which is a homology isomorphism in dimension n , as required.

6. Universal Coefficient Theorem

Consider complexes of abelian groups $(R = Z)$. If each K_n is free, the universal coefficient theorem for cohomology (Thm. III. 2.1) is an exact sequence

$$0 \to \operatorname{Ext}(H_{n-1}(K),G) \to H^n(K;G) \to \operatorname{Hom}(H_n(K),G) \to 0$$

We have a corresponding homology theorem:

Theorem 6.1. If K is a complex of torsion free abelian groups and A an abelian group, there is in each dimension n a split exact sequence of groups

$$0 \to H_n(K) \otimes A \xrightarrow{p} H_n(K;A) \to \mathrm{Tor}\,(H_{n-1}(K),A) \to 0$$

with both homomorphisms natural and p defined for cycles u of K and v of L by $p(\mathrm{cls}\ u \otimes \mathrm{cls}\ v) = \mathrm{cls}(u \otimes v)$.

This is a corollary to Thm. 5.4 with A the trivial complex $A_o = A$ and $A_n = 0$ for $n \neq 0$. Then $H_n(K;A)$ is the n^{th} homology group of the complex K over the abelian group A , that is $H_n(K \otimes A)$.

Corollary 6.2. For K and K' complexes of torsion free abelian groups while $f : K \to K'$ is a chain transformation with $f_* : H_n(K) \cong H_n(K')$ an isomorphism for all n , then $f_* : H_n(K;A) \to H_n(K';A)$ is an isomorphism for every abelian group A and all n .

The proof is similar to the proof of the corresponding cohomology result (Thm. III. 3.2) .

CHAPTER V

COHOMOLOGY OF GROUPS

The *integral group ring* $Z(\Pi)$ of an abstract group Π over the ring Z of integers is the set of all finite sums $\{\Sigma\, m_x x \ ; \ x \in \Pi \ , \ m_x \in Z\}$ with termwise addition and multiplication.

We consider $Z(\Pi)$-modules (Π-modules for short). Every abelian group A can be given the structure of a Π-module by defining a group homomorphism $\phi : \Pi \to \mathrm{Aut}\,(A)$, where $\mathrm{Aut}\,(A)$ denotes the group of automorphisms of A ; or equivalently by defining a function on $\Pi \times A$ to A, written xa, with

$$x(a_1 + a_2) = xa_1 + xa_2 \ ,$$
$$(x_1 x_2)a = x_1(x_2 a) \ , \ 1a = a \ .$$

In particular, any abelian group can be regarded as a *trivial* Π-module by taking $\phi(x) = 1$ or $xa = a$, for all x, a.

1. The Cohomology of a Group

The cohomology theory of a group Π over a group A is defined in terns of a standard "bar resolution" $B(Z(\Pi))$, a chain complex of Π-modules constructed from the original group Π. In this section we develop this theory and see that the cohomology

groups provide an example of the functors $\mathrm{Ext}^n(C,A)$, with $C = Z$ considered as a trivial Π-module. In the following sections we look at the cohomology groups in specific dimensions.

Let Π be a multiplicative group and $Z(\Pi)$ its group ring. Let B_n be the free Π-module with generators $[x_1,\ldots,x_n]$ all n-tuples of elements $x_1 \neq 1,\ldots,x_n \neq 1$ of Π. Set $[x_1,\ldots,x_n] = 0$ if any one $x_i = 1$, and call this the *normalization* condition. In particular B_o is free on one generator, denoted by $[\ ]$, so $B_o = Z(\Pi)[\ ] \cong Z(\Pi)$.

We define Π-module homomorphisms $\partial: B_n \to B_{n-1}$ for $n > 0$ by

$$\partial[x_1,\ldots,x_n] = x_1[x_2,\ldots,x_n] + \sum_{i=1}^{n-1}(-1)^i[x_1,\ldots,x_ix_{i+1},\ldots,x_n] + + (-1)^n[x_1,\ldots,x_{n-1}] . \tag{1.1}$$

For example $\partial_2[x,y] = x[y] - [xy] + [x]$ while $\partial_1[x] = x[\ ] - [\ ]$. We define an *augmentation* homomorphism $e : B_o \cong Z(\Pi) \to Z$ (Z, trivial module) by $e([\ ]) = 1$ $(e(\Sigma m_x x) = \Sigma m_x)$.

Then $B(Z(\Pi)) = \{B_n,\partial\}$ is a complex of Π-modules over (Z,e), termed the *bar resolution*. That $\partial^2 = 0$ may be proved directly but laboriously from (1.1) or from the following theorem.

Theorem 1.1. For any group Π the bar resolution $B(Z(\Pi))$ with augmentation e is a free resolution of the trivial Π-module Z.

Proof: The B_n are free by construction, so we must show that the sequence

$$\cdots \to B_n \xrightarrow{\partial} B_{n-1} \to \cdots \to B_1 \xrightarrow{\partial} B_o \xrightarrow{e} Z \to 0$$

is exact. We prove more: That the sequence is a complex of abelian groups with a *contracting homotopy* s ($s_{-1} : Z \to B_o$, $s_i : B_i \to B_{i+1}$, $i \geq 0$) given by $s_{-1} 1 := [\ ]$, $s_n x[x_1,\ldots,x_n] = [x,x_1,\ldots,x_n]$. s a contracting homotopy means that

$$es_{-1} = 1\ ,\ \partial_1 s_o + s_{-1}e = 1\ ,\ \partial_{n+1}s_n + s_{n-1}\partial_n = 1, \quad (n > 0)\ .$$

These equations are immediate from the definitons. By induction $e\partial_1 = 0$ and $\partial_n\partial_{n+1} = 0$ since $\partial_n\partial_{n+1}s_n = \partial_n(1 - s_{n-1}\partial_n)$ $= \partial_n - \partial_n s_{n-1}\partial_n = \partial_n - \partial_n + s_{n-2}\partial_{n-1}\partial_n = s_{n-2}\partial_{n-1}\partial_n$. Finally if $\partial_n x[x,x_1,\ldots,x_n] = 0$, then $s_{n-1}\partial_n x[x_1,\ldots,x_n] = 0$. Now $s_{n-1}\partial_n = 1 - \partial_{n+1}s_n$ and hence $\partial_{n+1}[x,x_1,\ldots,x_n] = x[x_1,\ldots,x_n]$. Thus $B(Z(\Pi))$ is a complex and a resolution of Z.

For any Π-module A we define the *cohomology groups* of Π over A as the cohomology groups of the complex $B(Z(\Pi))$ over A. Thus

$$H^n(\Pi;A) = H^n(B(Z(\Pi));A)\ .$$

Since B_n is free with generators $[x_1,\ldots,x_n]$ (no $x_i = 1$),

an n-cochain $f : B_n \to A$ is a Π-module homomorphism which is uniquely determined by its values on these generators. Thus the abelian group $B^n(\Pi;A)$ of n-cochains may be identified with the set of all those functions f of n arguments x_i in Π, with values in A, which satisfy the normalization conditions

$$f(x_1, \ldots, x_n) = 0 \ , \quad \text{if any} \quad x_i = 1 \ .$$

The coboundary homomorphism $\delta : \mathrm{Hom}\,(B_n, A) \to \mathrm{Hom}\,(B_{n+1}, A)$ is given by

$$\delta f(x_1,\ldots,x_{n+1}) = (-1)^{n+1} \{x_1 f(x_2,\ldots,x_{n+1}) \qquad (1.2)$$
$$+ \sum_{i=1}^{n} (-1)^i f(x_1,\ldots,x_i x_{i+1},\ldots,x_{n+1}) + (-1)^{n+1} f(x_1,\ldots,x_n)\}$$

For example a zero-dimensional cochain is a module homomorphism $f : Z(\Pi) \to A$ determined by its value $f(1) = a \in A$. It is a cocycle if $-\delta f(x) = f\partial(x) = f(x-1) = xa - a$ is zero. Hence the 0-cocycles correspond to the elements $a \in A$ which are invariant under all of Π, that is A^Π. Thus

$$H^o(\Pi;A) \cong A^\Pi \ .$$

Corollary 1.2. For any Π-module A, there is an isomorphism

$$\mathrm{Ext}^n\,(Z,A) \cong H^n(\Pi;A)$$

which is natural in A .

Since B is a free resolution of the trivial Π-module Z , the result is immediate from III . Thm. 4.5 .

For $E : A \rightarrowtail B \twoheadrightarrow C$ a short exact sequence of Π-modules, the corollary implies that we have an exact sequence of groups

$$\ldots \to H^n(\Pi;A) \to H^n(\Pi;B) \to H^n(\Pi,C) \xrightarrow{E_*} H^{n+1}(\Pi,A) \to \ldots \quad .$$

The connecting homomorphisms E_* are natural in E . For fixed Π , the cohomology groups $H^n(\Pi;A)$ are covariant functors of A which may be characterized, with the connecting homomorphisms, by three axioms like those for Ext (III.6) :

i. The above sequence is exact,

ii. $H^o(\Pi;A) \cong A^\Pi$,

iii. $H^n(\Pi;Q) = 0$, $n > 0$ and Q injective.

2. <u>The First Cohomology Group</u>

We describe $H^1(\Pi;A)$. A cochain f is a 1-cocycle if and only if for all $x , y \in \Pi$, $\delta f(x,y) = 0$, that is if and only if by (1.2) $f(xy) = x\,f(y) + f(x)$. Such a function f on a multiplicative group Π to a Π-module A is termed a *crossed homomorphism*. For example, if A is a trivial Π-module ($xa = a$ always) a crossed homomorphism is just an ordinary homomorphism from the multiplicative group Π to the additive group A . The group of

crossed homomorphism of Π into A is written $Z^1(\Pi;A)$.

f is a 1-coboundary if and only if there is a cochain $g \in \mathrm{Hom}\,(B_0,A)$, determined by some $a \in A$ with $g(1) = a$, such that $\delta g = f$. This is equivalent to $f(x) = xa - a$ for all x . Such f from Π to A are termed *principal crossed homomorphisms.* The group of all principal crossed homomorphisms from the group Π to the Π-module A is written $B^1(\Pi;A)$. Hence the first cohomology group of Π over A is the quotient group

$$H^1(\Pi;A) = Z^1(\Pi;A)/B^1(\Pi;A) \quad .$$

3. <u>Group Extensions</u>

A *group extension* is a short exact sequence

$$E : 0 \to G \xrightarrow{\kappa} B \xrightarrow{\sigma} \Pi \to 1 \tag{3.1}$$

of (not necessarilly abelian) groups; composition in 0 , A and B is usually written additively, while in Π and 1 it is written multiplicatively.

Conjugation in B yields a homomorphism $\theta : B \to \mathrm{Aut}\,(G)$ under which the action of each θb on any $g \in G$ is given by

$$\kappa[(\theta b)\, g] = b + \kappa g - b \ , \qquad b \ \in B \ , \ g \in G \ .$$

If $G = A$ abelian, then $\theta A = 1$, so that θ induces a homomorphism

$\phi : \Pi \to \mathrm{Aut}\ (A)$ with $\phi\sigma = \theta$. Thus ϕ is defined by

$$\kappa[(\phi\sigma b)\ a] = b + \kappa a - b \quad . \tag{3.2}$$

We then say that E is an extension of the abelian group A by the group Π with *operators* $\phi : \Pi \to \mathrm{Aut}\ (A)$.

The problem of group extensions is that of constructing all E , given A , Π and ϕ . Now ϕ gives A the structure of a Π-module; hence the group extension problem is that of constructing all E , given Π and a Π-module A . One such extension is the *semi-direct product* of A and Π whose elements are all pairs (a,x) with addition $(a,x) + (a_1,x_1) = (a + xa_1\ ,\ xx_1)$, $xa_1 = \phi(x)\ a_1$. That this is a group is immediate with the identity element $0 = (0,1)$ and inverse $-(a,x) = (-x^{-1}a, x^{-1})$. We have a short exact sequence

$$0 \to A \xrightarrow{\kappa} A \times_{\phi} \Pi \xrightarrow{\sigma} \Pi \to 1$$

with $\kappa a = (a,0)$, $\sigma(a,x) = x$. This splits under a right inverse ν for σ given by $\nu x = (0,x)$.

If E and E' are any two group extensions, a *morphism* $: E \to E'$ is a triple $\Gamma = (\alpha,\beta,\gamma)$ of group homomorphisms such that the diagram

$$\begin{array}{rcccccccc} E: & 0 & \to & A & \to & B & \to & \Pi & \to 1 \\ & & & \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & \\ E': & 0 & \to & A' & \to & B' & \to & \Pi' & \to 1 \end{array}$$

is commutative. If $\Gamma : E \to E'$ and $\Gamma' : E' \to E''$ are morphisms of extensions then so is $\Gamma'\Gamma : E \to E''$.

If E and E' are group extensions of the same module A by the group Π, a *congruence* $\Gamma : E \to E'$ is a morphism $\Gamma = (\alpha,\beta,\gamma)$ with $\alpha = 1_A$ and $\gamma = 1_\Pi$. The short five lemma shows β an isomorphism and hence each congruence has an iverse. We may therefore consider congruence classes of extensions. Let *Opext* (Π,A,ϕ) denote the set of congruence classes of extensions of the abelian group A by Π with operators ϕ. We wish to describe Opext .

Any extension which splits (under $\nu : \Pi \to B$) is congruent to the semi-direct product $A \times_\phi \Pi$ under $\beta : B \to A \times_\phi \Pi$ given by $\beta b = (\kappa^{-1}(b - \nu\sigma\, b) , \sigma b)$. For example, if Π is a free group with generators t_κ, then any epimorphism $\phi : B \to \Pi$ has a right inverse given by setting $\nu t_\kappa = b_\kappa$, where b_κ is any element in B with $\sigma(b_\kappa) = t_\kappa$. Hence Opext (Π,A,ϕ) with Π free consists of a single element.

As a more interesting case, let $\Pi = C_m(t)$, cyclic group of order m with generator t . Then there is a one-one correspondence

$$\text{Opext } (C_m(t), A, \phi) \longleftrightarrow [a;\ ta = a]/N_t A \quad ,$$

where $N_t : A \to A$ is a group homomorphism with $N_t a$ the *norm* of a , that is $N_t a = a + ta + \ldots + t^{m-1} a$. For details see MacLane [8] pp. 109-110 .

4. The Second Cohomology Group

The previous section suggests that Opext , like Ext and Tor , is a group. This group structure is described here by means of certain factor sets from which we see that Opext (Π,A,ϕ) is exactly the second cohomology group $H^2(\Pi;A)$.

Let E be an extension (3.1) in Opext (Π,A,ϕ) . For each $x \in \Pi$, choose a "representative" $u_x \in B$, $(\sigma u_x = x)$. In particular choose $u_1 = 0$. Thus if we consider A as a subgroup of B , each coset of A in B contains exactly one of u_x and each $b \in B$ can be uniquely written as $b = a + u_x$ for some $a \in A$. We write the operation $\phi(x)\, a = xa$; then (3.2) for $b = u_x$ becomes

$$xa = u_x + a - u_x \quad . \tag{4.1}$$

On the other hand the sum $u_x + u_y$ lies in the same coset as u_{xy} , so there are unique elements $f(x,y) \in A$ with

$$f(x,y) = u_x + u_y - u_{xy} \tag{4.2}$$

since $u_1 = 0$, we clearly have

$$f(x,1) = 0 = f(1,y) \quad . \tag{4.3}$$

The function f is called a *factor set* of the extension E .

It follows from (4.1) and (4.2) that

$$(u_x + u_y) + u_z = f(x,y) + f(xy,z) + u_{xyz}$$

$$u_x + (u_y + u_z) = xf(y,z) + f(x,yz) + u_{xyz}$$

and the associativity in B gives

$$xf(y,z) + f(x,yz) = f(x,y) + f(xy,z) \tag{4.4}$$

f depends on a choice of representatives; if u'_x is a second set with $u'_1 = 0$, there exists a function $g : \Pi \to A$ with $g(1) = 0$ and $g(x) = u'_x - u_x$. In such a case we would have

$$f'(x,y) = \delta g(x,y) + f(x,y)$$

where δg is a function given by

$$\delta g(x,y) = xg(y) - g(xy) + g(x) \ . \tag{4.5}$$

Then δg satisfies (4.4) with f replaced by δg .

The functions f on $\Pi \times \Pi$ to A which satisfy the identity (4.4) and the normalization conditions (4.3) are the 2-cocycles of the complex Hom (B,A) , where B is the bar resolution: For, a 2-cochain f of Hom (B,A) is a Π-homomorphism $f : B_2 \to A$; it is determined by its values $f[x,y]$ on the free generators of

B_2 ; hence is in effect a function on $\Pi \times \Pi$ to A with $f(x,1) = 0 = f(1,y)$. The condition that f be a cocycle can be seen by (1.2) to be simply (4.4) . Similarly f is a 2 dimensional coboundary if and only if there is a 1-cochain g with $\delta g = f$. This by (1.2) is just (4.5) .

Hence the second cohomology group of Π over A may be written as the quotient

$$H^2(\Pi;A) = Z^2(\Pi;A)/B^2(\Pi;A)$$

where $Z^2(\Pi;A)$ is the group of 2-cocycles of Hom (B,A) or the group of functions f on $\Pi \times \Pi$ to A satisfying (4.3) and (4.4) , while $B^2(\Pi;A)$ is the group of 2-coboundaries or the subgroup of $Z^2(\Pi;A)$ consisting of all functions f of the form $f = \delta g$ where δg is defined by (4.6) with $g(1) = 0$.

The set Opext may be considered as a group by the following theorem.

Theorem 4.1. Given $\phi : \Pi \to \mathrm{Aut}\ (A)$, A abelian, the function ω which assigns to each extension of A by Π with operators ϕ the congruence class of one of its factor sets is a 1 - 1 correspondence

$$\omega : \mathrm{Opext}\ (\Pi,A,\phi) \longleftrightarrow H^2(\Pi;A)$$

between the set Opext of all congruence classes of such extensions

and the two dimensional cohomology group. Under this correspondence the semi-direct product corresponds to $Z^2(\Pi;A)$.

Proof. Since the factor set of an extension is well defined modulo the subgroup B^2 , and since congruent extensions have the same factor sets, we know that the correspondence is well defined. The semi-direct product clearly has the trivial function $f(x,y) = 0$ as one of its factor sets. If two extensions yield factor sets whose difference is some function δg , then a change of representatives in one extension will make the factor sets equal and the extensions congruent.

Finally to show that ω is onto we define a group B to consist of all pairs (a,x) with addition given by

$$(a,x) + (a_1,y) = (a + xa_1 + f(x,y), xy) \quad .$$

This is clearly a group and yields an extension with representatives $u_x = (0,x)$ and factor set f .

5. Cohomology of Finite Groups

For Π finite the coboundary formula (1.2) gives the following result.

Proposition 5.1. If Π is a finite group of order k , every element of $H^n(\Pi;A)$ for $n > 0$ has order dividing k .

Proof. For each n-cochain f we define an $(n - 1)$ cochain g by

$$g(x_1, \dots, x_{n-1}) = \sum_{x \in \Pi} f(x_1, \dots, x_{n-1}, x) .$$

Add the identities (1.2) for all $x = x_{n+1}$ in Π. The last term is independent of x; in the next to the last term, for x_n fixed,

$$\sum_x f(\dots, x_{n-1}, x_n x) = \sum_x f(\dots, x_{n-1}, x) = g(\dots, x_{n-1}) .$$

Hence the result is

$$\sum_{x \in n} \delta f(x_1, \dots, x_n, x) = -\delta g(x_1, \dots, x_n) + kf(x_1, \dots, x_n) .$$

For f an n-cocycle $\delta f = 0$. This gives $kf = \delta g$ a coboundary, hence the result.

Corollary 5.2. If Π is finite, D a divisible torsion free abelian group and in some way a Π-module, then $H^n(\Pi;D) = 0$ for $n > 0$.

Proof. For g as above, there is an $(n - 1)$-cochain h with $g = kh$. Then $kf = \pm\, k\delta h$; since D has no elements of finite order, $f = \pm\, \delta h$, and the cocycle f is a coboundary.

Corollary 5.3. If Π is finite, P the additive group of real numbers, mod 1, and P and Z trivial Π-modules, then $H^2(\Pi;Z) \cong \mathrm{Hom}(\Pi,P)$.

Proof. The additive group R of real numbers is a divisible torsion free group. The short exact sequence $Z \rightarrowtail R \twoheadrightarrow P$ of trivial Π-moduels yields the exact sequence

$$H^1(\Pi;R) \to H^1(\Pi;P) \to H^2(\Pi;Z) \to H^2(\Pi;R) \quad .$$

By Cor. 5.2 the two outside groups vanish and since P has trivial Π-module structre $H^1(\Pi;P) = \mathrm{Hom}(\Pi,P)$. Hence the result.

The (abelian) group $\mathrm{Hom}(\Pi,P)$ of all group homomorphisms $\Pi \to P$ is known as the *character* group of Π.

6. Cohomology of Cyclic Groups

Since $H^n(\Pi;A) = \mathrm{Ext}^n(Z,A)$, we may calculate the cohomology of a particular group Π by using a Π-module projective resolution of Z suitably adapted to the structure of the group Π.

Let $\Pi = C_m(t)$ be the cyclic group of order m with generator t. Its group ring $\Gamma = Z(C_m(t))$ is the ring of polynomials $u = \sum_{i=o}^{m-1} a_i t^i$ with $a_i \in A$ and $t^m = 1$. Two particular elements in Γ are

$$N = 1 + t + \ldots + t^{m-1} \ , \ D = t - 1 \ .$$

From this we can form the sequence

$$W : \ldots \to \Gamma \xrightarrow{D_*} \Gamma \xrightarrow{N_*} \Gamma \xrightarrow{D_*} \Gamma \xrightarrow{e} Z \to 0$$

with homomorphisms given by $D_*u = Du$, $N_*u = Nu$ and e the augmentation. It follows straightforwardly that the above sequence is a free resolution of Z .

For any Γ-module A there is an isomorphism $\text{Hom}\,(\Gamma,A) \cong A$ under $f \longleftrightarrow f(1)$. Hence the cochain complex $\text{Hom}\,(W,A)$, with the usual signs $\delta f = (-1)^{n+1} f\partial$ for the coboundary, becomes

$$0 \to A \xrightarrow{e^*} A \xrightarrow{-D^*} A \xrightarrow{N^*} A \xrightarrow{-D^*} A \to \ldots$$

with $N^*a = Na$, $D^*a = Da = (t - 1)a$. The cohomology groups of $C_m(t)$ are those of this cochain complex, hence

Theorem 6.1. For a finite cyclic group $C_m(t)$ of order m and generator t , and any $C_m(t)$-module A , the cohomology groups are

$$H^0(C_m;A) = [a;ta = a]$$

$$H^{2n}(C_m;A) = [a;ta = a]/N^*A \qquad n > 0$$

$$H^{2n+1}(C_m;A) = [a;Na = 0]/D^*A \qquad n \geq 0 \qquad .$$

7. The Third Cohomology Group (Obstructions to Extensions)

The 3-dimensional cohomology groups appear in the study of extensions of a non-abelian group G .

For $h \in G$, denote by μ_h the *inner automorphism* $\mu_h g = h + g - h$. Then $\mu : G \to \mathrm{Aut}\,(G)$ is a group homomorphism and $\mu(G) = \mathrm{In}\,(G)$ which is a normal subgroup of $\mathrm{Aut}\,(G)$, as one sees by observing that for $\eta \in \mathrm{Aut}\,(G)$

$$\eta\, \mu_h\, \eta^{-1} = \mu_{\eta h} \tag{7.1}$$

The factor group $\mathrm{Aut}\,(G)/\mathrm{In}\,(G)$ is termed the group of *automorphism classes* of G . The kernel of μ is the center C of G and we therefore have an exact sequence

$$0 \to C \to G \xrightarrow{\mu} \mathrm{Aut}\,(G) \to \mathrm{Aut}\,(G)/\mathrm{In}\,(G) \to 1 \tag{7.2}$$

Any group extension

$$E : 0 \to G \xrightarrow{\kappa} B \xrightarrow{\sigma} \Pi \to 1$$

of G by Π determines, via conjugation in B , a homomorphism $\theta : B \to \mathrm{Aut}\,(G)$ for which $\theta(\kappa G) \subseteq \mathrm{In}\,(G)$. It hence determines an induced homomorphism $\psi : \Pi \to \mathrm{Aut}\,(G)\,/\,\mathrm{In}\,(G)$. We say that E has conjugation class ψ . Conversely we call a pair of groups Π and G , together with a homomorphism $\psi : \Pi \to \mathrm{Aut}\,(G)/\mathrm{In}\,(G)$ an

abstract kernel. The general problem of group extensions is that of constructing all extensions E to a given abstract kernel (Π, G, ψ).

A given extension E may be described as follows. Consider G as a subgroup of B. To each $x \in \Pi$ choose representatives $u_x \in B$ with $\sigma u_x = x$, in particular choosing $u_1 = 0$. Conjugation by u_x yields an automorphism $\phi(x) \in \psi(x)$ of G with

$$\phi(x)\, g + u_x = u_x + g \quad . \tag{7.3}$$

The sum $u_x + u_y$ equals u_{xy} up to a summand in G, which we may denote as $f(x,y) \in G$.

$$f(x,y) = u_x + u_y - u_{xy} \quad .$$

The associativity law for $u_x + u_y + u_z$ yields

$$\phi(x)\, f(y,z) + f(x,yz) = f(x,y) + f(xy,z) \; . \tag{7.4}$$

Conjugation by the left and right hand side of (7.3) gives

$$\phi(x)\, \phi(y) = \mu f(x,y)\, \phi(xy) \quad . \tag{7.5}$$

Thus μf measures the deviation of ϕ from a homomorphism $\Pi \to \mathrm{Aut}\,(G)$.

Conversely, these conditions may be used to construct an extension

as follows.

Lemma 7.1. Given Π , G and functions $\phi : \Pi \to \mathrm{Aut}\,(G)$ and $f : \Pi \times \Pi \to G$ satisfying (7.4) and (7.5) and the conditions $\phi(1) = 1$ and $f(x,1) = 0 = f(1,y)$. The set $B_o[G,\phi,g,\Pi]$ of all pairs (g,x) under the sum defined by

$$(g,x) + (g_1,y) = (g + \phi(x)g_1 + f(x,y),\ xy)$$

is a group. With the homomorphisms $g \to (g,1)$ and $(g,x) \to x$, $G \rightarrowtail B_o \twoheadrightarrow \Pi$ is an extension of G by Π with conjugation class given by the automorphism class of ϕ .

The proof is immediate noting that $(0,1)$ is the zero element while the inverse of (g,x) is $(-f(x^{-1},x) - \phi(x^{-1})g,\ x^{-1})$. The group B_o so constructed is termed a *crossed product* group. Our analysis prior to the lemma showed that any extension is isomorphic to such a crossed product in the following explicit sense.

Lemma 7.2. If $\phi(x) \in \psi(x)$ has $\phi(1) = 1$, then any extension E of the abstract kernel (Π,G,ψ) is congruent to a crossed product extension $[G,\phi,f,\Pi]$ with the given function ϕ .

Proof. If $E = (\kappa,\sigma)$ we can choose representatives u_x so that $g \to u_x + g - u_x$ is the automorphism $\phi(x)$ in the class $\psi(x)$. Any $b \in B$ can be written uniquely as $b = g + u_x$, so we define

$\beta : B \to B_o$ by $\beta b = (g,x)$ which satisfies the requirements and shows $E \equiv B_o$.

Suppose now that just the abstract kernel (Π,G,ψ) is known. In each class $\psi(x)$ choose an automorphism $\phi(x)$ satisfying $\phi(1) = 1$. Since $\psi : \Pi \to \mathrm{Aut}\,(G)/\mathrm{In}\,(G)$ is a homomorphism, that is $\psi(xy) = \psi(x)\,\psi(y)$, we have $\phi(x)\,\phi(y)\,\phi(xy)^{-1} \varepsilon \mathrm{In}\,(G)$. Hence there are elements $f(x,y) \varepsilon G$, $(f(x,1) = 0 = f(1,y))$, with $\phi(x)\,\phi(y) = \mu f(x,y)\,\phi(xy)$. This is equation (7.5) ; we would like (7.4) to hold also, but this need not be so. The associative law for composition of functions, $\phi(x)(\phi(y)\,\phi(z)) = (\phi(x)\,\phi(y))\,\phi(z)$, shows that (7.4) holds only after μ is applied to both sides. The kernel of μ is the center C of G ; hence there is for all x, y, z an element $k(x,y,z) \varepsilon C$ such that

$$\phi(x)f(y,z) + f(x,yz) + k(x,y,z) = f(x,y) + f(xy,z) \qquad (7.4)'$$

Clearly $k(x,y,1) = 0 = k(x,1,z) = k(1,y,z)$ and hence k may be considered as a normalized 3-cochain of Π over C .

The abelian group C may be regarded as a Π-module, for each automorphism $\phi(x)$ of G carries C into C and yields for $c \varepsilon C$ an automorphism $c \to \phi(x)\,c$ independent of the choice of $\phi(x)$ in its class $\psi(x)$. We may thus write $\phi(x)\,c$ as $x\,c$.

We call the cochain k of (7.4)' an *obstruction* of the abstract

kernel (Π, G, ψ) . There are various obstructions to a given kernel depending on the choice of $\phi(x) \in \psi(x)$ and of f satisfying (7.5) , but when there is one we have shown in (7.4) that there is an obstruction $k = 0$; hence

Lemma 7.3. An abstract kernel (Π, G, ψ) has an extension if and only if one of its obstructions is the cochain identically 0 .

Next we prove

Lemma 7.4. Any obstruction k of a kernel (Π, G, ψ) is a 3-dimensional cocycle of $B(Z(\Pi))$.

Proof. We must show that $\delta_3 k = 0$, that is $k\partial_4 = 0$. This becomes immediate upon calculating for x, y, z, t in Π the expression

$$L = \phi(x)\ (\phi(y)\ f(z,t) + f(y,zt)) + f(x,yzt)$$

in two ways. In the first apply (7.4)' to the inside terms and then apply $\phi(x)$. In the second rewrite $\phi(x)\ \phi(y)$ as $\mu f(x,y)\,\phi(xy)$ and then use (7.4)' .

The effect of choosing different ϕ and f in the construction of an obstruction is given by the next two lemmas.

Lemma 7.5. For given $\phi(x) \in \psi(x)$, a change in the choice of f

in (7.5) replaces k by a cohomologous cocycle.

Proof. Since the kernel of μ is the center C of G, any other choice of f in (7.5) must have the form

$$f'(x,y) = h(x,y) + f(x,y)$$

where $h(x,1) = 0 = h(1,y)$. Thus h may be viewed as a 2-dimensional normalized cochain of Π with values in C. On calculating the obstruction k' obtained from f' we get $k' - k = h$, and so k and k' are cohomologous cocycles.

Since h is arbitrary, k may actually be replaced by any cohomologous cocycle.

Lemma 7.6. By choosing a different $\phi(x) \in \psi(x)$, we may select a suitable f such that the obstruction k is unaltered.

Proof. Replace $\phi(x)$ by $\phi'(x)$ in $\psi(x)$ with $\phi'(1) = 1$. $\phi(x)$ and $\phi'(x)$ lie in the same class and hence there is an element $g(x) \in G$ with $g(1) = 0$ such that $\phi'(x) = \mu g(x)\,\phi(x)$. Then by (7.1) and (7.5)

$$\phi'(x)\,\phi'(y) = \mu(g(x) + \phi(x)g(y) + f(x,y) - g(xy))\,\phi'(xy) .$$

A new function $f'(x,y)$ may be given by the portion above in

brackets. This can be written as

$$f'(x,y) + g(x,y) = \phi'(x)\ g(y) + g(x) + f(x,y) .$$

A series of calculations will now yield

$$\begin{aligned} &\phi'(x)\ f'(y,z) + f'(x,yz) + g(xyz) \\ &\qquad = -\ k(x,y,z) + f'(x,y) + f'(xy,z) + g(xyz) , \end{aligned}$$

which upon deletion of the term $g(xyz)$ shows the obstruction k to be unaltered when $\phi(x)$ is .

These results may be summarized as follows.

Theorem 7.7. In the abstract kernel (Π,G,ψ) interpret the center C as a Π-module with operators $xc = \phi(x)\ c$ for any choice of automorphisms $\phi(x) \varepsilon \psi(x)$. The assignment to this kernel of the cohomology class of any one of its obstructions yields a well defined object Obs $(\Pi,G,\psi) \varepsilon H^3(\Pi;C)$. The kernel has an extension if and only if Obs = 0 .

To complete the study of the extension problem, we mention the following result on the manifold of extensions. The details can be found in MacLane [8] pp. 128-129 .

Theorem 7.8. If the abstract kernel (Π,G,ψ) has an extension, then the set of congruence classes of the extensions is in 1-1

correspondence with the set $H^2(\Pi;C)$ where C is the center of G with module structure as in Thm. 7.7 .

8. The Third Cohomology Group (Realization of Obstructions)

We have shown that the obstruction to the extension problem is an element of $H^3(\Pi;C)$. If $C = 0$ the obstruction vanishes and hence the extension problem has a solution. The result is

Lemma 8.1. If the group G has center C , then any abstract kernel (Π,G,ψ) has an extension.

Proof. (7.2) reduces to $E : G \rightarrowtail \mathrm{Aut}\,(G) \twoheadrightarrow \mathrm{Aut}\,(G)/\mathrm{In}\,(G)$. Then $E\psi$ is the desired extension.

We now show that every 3-cocycle can be realized as an obstruction. This will give us a many-one correspondence of abstract kernels with center C to the group $H^3(\Pi;C)$.

Theorem 8.2. Given a group Π , a Π-module C , and any element $\mathrm{cls}\ k \in H^3(\Pi;C)$, there exists a group G with center C and a homomorphism $\psi : \Pi \to \mathrm{Aut}\,(G)/\mathrm{In}\,(G)$ inducing the given Π-module structure on C and such that $\mathrm{Obs}\,(\Pi,G,\psi) = \mathrm{cls}\ k$.

Proof. Part I. Π not of order 2 . Let G be the direct product $C \times F$ where C is the given Π-module and F is a free group with generators all symbols $[x,y]$ for $x, y \neq 1$ in Π .

Define a function f on $\Pi \times \Pi$ to G by $f(x,y) = (0,[x,y])$ for $x, y \neq 1$ and $f(x,1) = 0 = f(1,y)$. For such $x \varepsilon \Pi$ define an endomorphism $\beta(x) : G \to G$ by

$$\beta(x)(c,1) = (xc,1) \tag{8.1}$$

$$\beta(x)(0,[y,z]) = (k(x,y,z),1) + (0,[x,y]) + (0,[xy,z]) - (0,[x,yz]) .$$

This also holds for $[y,z]$ replaced by $f(y,z)$, that is with y or $z = 1$.

By this definiton we see that $\beta(1)$ is the identity automorphism. It can also be seen that

$$\beta(x)\ \beta(y) = \mu f(x,y)\ \beta(xy) : G \to G \quad . \tag{8.2}$$

This is done by applying both sides to $c = (c,1)$ and generator $[z,t] = (0,[z,t])$ in G .

$\beta(x)$ is actually an automorphism of G : For any x , $\beta(x)\ \beta(x^{-1}) = \mu f(x,x^{-1})\ \beta(xx^{-1}) = \mu f(x,x^{-1})\ \varepsilon\ \mathrm{In}\ (G)$, hence $\beta(x)$ is onto; similarly it is one-one.

Let $\psi(x)$ denote the automorphism class of G containing $\beta(x)$. By (8.2) ψ is a homomorphism $\Pi \to \mathrm{Aut}\ (G)/\mathrm{In}\ (G)$, hence (Π,G,ψ) is an abstract kernel. As Π contains more than two elements, the free group F has more than one generator, hence is centerless, so that C is actually the center of $G = C \times F$. Then the element

Obs $(\Pi,G,\psi) \in H^3(\Pi;C)$ that we can obtain by methods of the previous section is the given element cls k, because of the conditions (8.1) and (8.2) used in our construction.

Part II. Π cyclic of order 2. Let $\Pi = \{1,s;s^2 = 1\}$. The previous argument fails here as the F constructed would be abelian. In this case a (normalized) 3-cocycle is determined by a single element $k(x,x,x) = c_o \in C$, and the condition $k = 0$ is equivalent to $sc_o = -c_o$.

To construct G, let R be the multiplicative group of positive rationals. It is generated by the prime numbers, which we arrange in an infinite list $\ldots, p_{-2}, p_{-1}, p_o, p_1, p_2, \ldots$ without ends. Let L be the direct product $C \times R$ and define an automorphism β of L by setting

$$\beta(c,1) = (sc,1) , \quad \beta(0,p_i) = (0,p_{i+1}) .$$

Construct G as an extension of an infinite cyclic group J, with generator t, by L, that is let

$$G = \{\ell + nt \; ; \; \ell \in L , \quad n \in Z\} .$$

Under addition given by $t + \ell = \beta^2\ell + t$, G becomes a group.

The only elements of L invariant under β^2 are the elements of C; from this it follows that the center of G is C. The automorphism β can be extended to an automorphism ϕ of G by

setting $\phi(\ell + t) = \beta\ell + (c_o,1) + t$. Then $\phi^2(\ell + nt) = \beta^2\ell + nt$. Furthermore, the automorphism ϕ^2 is simply conjugation by t and hence ϕ is an automorphism. Therefore the equation $\psi(1) = 1$ and $\psi(s) = \phi$ determine a homomorphism $\psi : \Pi \to \mathrm{Aut}\,(G)/\mathrm{In}\,(G)$, and (Π,G,ψ) is an abstract kernel with center C .

In the construction of § 7 we may choose $\phi \in \psi(s)$ as above and pick $f(s,s) = t = 0 + t$. Then $k'(s,s,s) = \phi f(s,s) - f(s,s) = (c_o,1)$, and so $k = k'$. This completes the proof in the special case.

This many-one correspondence established here can be so decorated as to become a group isomorphism; one first defines a relation of similarity between abstract kernels such that two kernels are similar if and only if they have the same obstruction; with a suitable product of kernels the group of similarity classes of kernels (Π,G,ψ) with fixed Π and fixed Π-module C as center is then isomorphic to $H^3(\Pi;C)$. The details are given in Eilenberg-MacLane [1947] .

No reasonable analogous interpretation of $H^4(\Pi;C)$ or of higher dimensional cohomology groups is known.

CHAPTER VI

APPLICATIONS OF THE COHOMOLOGY OF GROUPS

We now use some of the results of the cohomology theory of groups to first establish a fundamental result in group theory and then to look at the "equivariant" cohomology of a topological space.

1. Schur's Theorem

We prove here the Schur-Zassenhaus Lemma which says that any extension of a finite group by another finite group, whose orders are relatively prime, splits. We prove the special abelian case separately as it involves a different part of the theory. The chief results from general group theory used in the proof of the theorem in its general form are stated below for reference in the proof.

Lemma 1.1. Any finite p-group with more than one element has non-trivial center.

Lemma 1.2. Any two maximal p-subgroups of a finite group are conjugate.

Theorem 1.3. If the finite abelian groups A and C have

relatively prime order, then every extension of A by C splits.

Proof. Let A and C have orders m and n respectively, and let $\mu_m : C \to C$ be the homomorphism given by $\mu_m c = mc$. Since m and n are relatively prime, there is an m' with $mm' \equiv 1 \pmod{n}$; hence μ_m is an automorphism, and every element of Ext (C,A) has the form $E\mu_m$ for some E. But $\mu_m = 1_C + \dots + 1_C$, with m-summands, so

$$E\mu_m = E(1_C + \dots + 1_C) \equiv (1_A + \dots + 1_A)E = \nu_m E = 0 ,$$

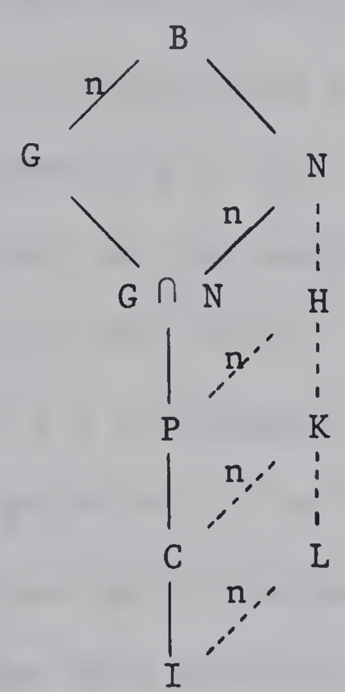
where $\nu_m : A \to A$ is $\nu_m(a) = ma = 0$. This completes the proof as the zero element of the group Ext (C,A) is the split sequence.

Theorem 1.4. If the integers m and n are relatively prime, any extension of a group of order m by one of order n splits.

Proof. Let $E : G \rightarrowtail B \overset{\sigma}{\twoheadrightarrow} \Pi$ be such an extension with G of order m and Π of order n. This extension splits if σ has a right inverse; that is, if B contains a subgroup of order n mapped isomorphically onto Π.

If G is abelian the given extension is by (V. Thm. 4.1) an element $t \in H^2(\Pi;G)$. By (V. Prop. 5.1) $nt = 0$; trivially $mt = 0$. Since m and n are relatively prime, $t = 0$. Thus E is the semi-direct product extension.

For G non-abelian the proof is by induction on the order m of G. Take a prime p dividing m and a maximal p-subgroup P of B. The normalizer N of P in B is defined to be the set of all b with $bPb^{-1} = P$. The index $[B:N]$ is then the number of conjugates of P in B. All of these conjugates must be in G and are maximal p-subgroups there. By Lemma 1.2 they are all conjugate in G. Now $G \cap N$ is the normalizer of P in G, so the index $[G : G \cap N]$ is the number of these conjugates and is therefore equal to $[B : N]$. This index equality (see diagram) proves also that $n = [B : G] = [N : G \cap N]$. Now P and $G \cap N$ are normal subgroups of N, and N/P is an extension of the group $(G \cap N)/P$, of order some proper divisor of m, by the group $N/G \cap N$ of order n. By the induction assumption, N/P thus contains a subgroup of order n, which may be written in the form H/P for some H with $P \subseteq H \subseteq N$ and $[H : P] = n$.



The center C of P is, by Lemma 1.1 not 1. Conjugation by elements of $H \subseteq N$ maps P onto itself and hence C onto itself, so that C and P are normal in H. Thus H/C is an extension of the p-group P/C by the group H/P of order n prime to p. Since $C \neq 1$, the order of P/C is less than m, so the induction assumption provides a subgroup $K/C \subseteq H/C$ of order n. This group K is an extension of the abelian p-subgroup C by a group K/C of order n,

hence splits by the abelian case already proven. This splitting provides a subgroup $L \subseteq K$ of order n, and the subgroup L splits the original extension B.

2. Spaces with Operators

For any topological space X, let $A(X)$ denote the group of homeomorphisms of X. A group Π *operates* on X if there is a homomorphism $\mu : \Pi \to A(X)$. Equivalently, to each $a \in \Pi$ and $x \in X$, there is a unique point $ax = \mu(a)x \in X$ such that ax is continuous in x for fixed a and always $(a_1a_2)x = a_1(a_2x)$ and $1x = x$. An open set U in X is called *proper* if $aU \cap U = \phi$, whenever $a \neq 1$. The group Π operates *properly* on X if every point of X is contained in a proper open subset of X; then every open set in X is the union of proper open sets, so that the proper open sets constitute a basis for the topology of X.

We assume hence forth that Π operates properly on X. By an *orbit* of a point x under the action of Π we mean $\{ax ; a \in \Pi\}$. Each x lies in exactly one orbit, so that the orbits provide a disjoint subdivision of X. Let X/Π be the quotient space whose elements are the orbits of X. Let $p : X \to X/\Pi$ be the usual projection of a point x to its orbit px. Thus $px_1 = px_2$ if and only if there is an $a \in \Pi$ such that $ax_1 = x_2$. The topology of X/Π is defined as having a basis consisting of those sets pU where U is open proper in X. The sets $V = pU$ are called *proper* in X/Π.

The above discussion gives us the following proposition.

Proposition 2.1. The map $p : X \to X/\Pi$ is continuous. The space X/Π is covered by proper open sets V ; each $p^{-1}V$ is the union of disjoint open sets aU for $a \in \Pi$, and the restriction $p|aU$ is a homeomorphism $aU \cong V$.

This proposition states that X is a "covering space" for X/Π under the map p . The aU are the *sheets* of X over V .

We now consider the singular homology of X , as defined in Chap. II .

Lemma 2.2. If the group Π operates properly on the space X , then the singular complex $C(X)$ is a complex of free Π-modules.

Proof. The group $C_n(X)$ of n-chains is the free abelian group generated by the singular n-simplices $T : \Delta^n \to X$. For each $a \in \Pi$, the composite aT is also a singular n-simplex; the operators $T \to aT$ makes $C_n(X)$ a Π-module. If Te^i denotes the i^{th} face of T , then for $\partial T = \sum(-1)^i \, Te^i$, $\partial(aT) = a(\partial T)$, so that ∂ is a Π-module homomorphism. Thus $C(X)$ is a complex of Π-modules. To show C_n free, pick any subset $X_o \subseteq X$ containing exactly one point from each orbit of X under Π . Then those singular n-simplices T with initial vertex in X_o constitutes a set of free generators for $C_n(X)$ as a module.

Lemma 2.3. If the group Π operates properly on the space X, any $T : \Delta^n \to X/\Pi$ can be written as $T = pT'$ for some $T' : \Delta^n \to X$. With suitable choice of one T' for each T, those T' are free generators of the Π-module $C_n(X)$.

Proof. If T is "small" in the sense that $T(\Delta^n)$ is contained in a proper open subset V of X/Π, and if U is any sheet over V then T can be lifted to $T' = (P|U)^{-1} T$ in U. The general case can then be handled by subdividing Δ^n into small pieces, lifting T in succession on there pieces. As Δ^n is homeomorphic to the unit n-cube I^n, it suffices to lift any $T : I^n \to X/\Pi$. The cube I^n is covered by the inverse images $T^{-1}(V)$ of proper open sets of X/Π. Since I^n is a compact metric space, the Lebesque Lemma provides a real $e > 0$ such that any subset of diameter less than e. Then T can be lifted in succession on the cubes of this subdivision, beginning with the cubes on the bottom layer. When we came to lift T on any one cube, the continuous lifting T' will already be defined on a certain connected set of faces of this cube and will lie in one sheet U over some proper V; the rest of the cube is then lifted by $(p|U)^{-1}$. This completes the proof.

Proposition 2.4. If Π operates properly on X, while the abelian group A has the trivial Π-module structure, then $p : X \to X/\Pi$ induces an isomorphism $p^* : \mathrm{Hom}\,(C(X/\Pi), A) \cong \mathrm{Hom}\,(C(X), A)$ of chain complexes and hence an isomorphism

$$p^* : H^n(X/\Pi ; A) \cong H^n(\mathrm{Hom}\,(C(X), A)) \quad .$$

Proof. A cochain $f : C_n(X/\Pi) \to A$ is uniquely determined by its values on the n-simplices T of X/Π, while a cochain f' of $C(X)$, as a module homomorphism $f' : C_n(X) \to A$, is uniquely determined by its values on the free module generators T' of $C_n(X)$. Since these generators are in 1-1 correspondence $T' \to pT'$ by Lemma 2.3 and since $(p^* f)T' = f(pT')$, the result follows.

More generally, when A is any Π-module, the homology of $\mathrm{Hom}\,(C(X),A)$ is known as the *equivariant cohomology* of X over A.

A topological space X is said to be *acyclic* if its homology groups $H_n(X) = 0$ for $n \neq 0$ while $H_o(X) \cong Z$. For example, a one point space is acyclic. The main result now is

Theorem 2.5. If a group Π operates properly on an acyclic space X, and if A is an abelian group with trivial Π-module structure, there is an isomorphism

$$H^n(X/\Pi;A) \cong H^n(\Pi;A) \,, \qquad n = 0, 1, \ldots$$

natural in A, between the cohomology groups of the quotient space X/Π and those of the group Π.

Proof. $H_o(X) \cong Z$ yields an epimorphism $C_o(X) \to Z$ with kernel $C_1(X)$. Thus the exact sequence of Π-modules

$$\cdots \to C_2(X) \to C_1(X) \to C_o(X) \to Z \to 0$$

is a free resolution of the trivial module Z . Hence the equivariant cohomology of X over A is $\mathrm{Ext}^n(Z,A)$, which by (V. Cor. 1.2) is $H^n(\Pi;A)$. The result follows from Prop. 2.4 .

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